Chapter 1

Asymptotic problems for nonlinear elliptic equations

1.1 Nonlinear elliptic boundary-value problems in unbounded domains and the asymptotic behavior of their solutions.

In this section we consider the problem of existence, uniqueness and asymptotic properties at infinity of solutions of boundary-value problems in unbounded domains for a class of nonlinear second order elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - a(x)|u|^{p-1}u = f(x). \tag{1}$$

Equations of this type were studied in many papers (see, for example, [1]–[4]), [9]–[27].

Let Ω be a smooth unbounded domain in $\mathbf{R}^n = (x_1, \ldots, x_n)$. In particular, Ω can be all of $\mathbf{R}^n$. We consider the boundary-value problem for the equation (1) in Ω, with the boundary conditions

$$u = 0 \text{ on } \Gamma_D, \qquad \frac{\partial u}{\partial \mu} = 0 \text{ on } \Gamma_N, \tag{2}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$, and Γ_D or Γ_N can be the empty set on $\partial\Omega$. We assume that $a_{ij} \in L^\infty_{\text{loc}}(\Omega)$, and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \qquad x \in \Omega,\ \xi \in \mathbf{R}^n,$$

$\lambda = const > 0$, $p > 1$, $a \in L^1_{\rm loc}(\Omega)$ and $a(x) \geqslant a_0 = const > 0$, for any $x \in \Omega$. Here as usual

$$\frac{\partial u}{\partial \mu} \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_j ,$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal vector to $\partial\Omega$.

We introduce the following notations:

$$\begin{aligned} &B_R = \{x \ : \ |x| < R\},\ \Omega_R = \Omega \cap B_R, \\ &V(\Omega_R) = \{w \ : \ w \in H^1(\Omega_R),\ w = 0 \text{ on } \Gamma_D \cap B_R\}, \\ &V_{\rm loc}(\Omega) = \{w \ : \ w \in V(\Omega_R) \text{ for any } R > 0\}. \end{aligned}$$

We also consider the dual spaces $V^*(\Omega_R)$ and $V^*_{\rm loc}(\Omega)$ respectively. Let $f \in V^*_{\rm loc}(\Omega)$. The space $H^1(G)$ is a Hilbert space with the scalar product $\int_G (uv + \nabla u \cdot \nabla v) dx$.

A function $u \in V_{\rm loc}(\Omega)$ is called a weak solution of problem (1),(2), if $a(x)|u|^{p-1}u \in L^1_{\rm loc}(\Omega)$ and

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \mathrm{d}x + \int_\Omega a(x)|u|^{p-1}u\varphi \mathrm{d}x = - < f, \varphi > \qquad (3)$$

for any $\varphi \in V_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$, which has a compact support in $\Omega \cup \Gamma_N$.

Let us introduce the cut-off function

$$\Theta_R(x) = \theta^2 \left(\frac{|x|}{R} \right) \quad \text{for } R > 0,$$

where $\theta \in C^\infty(\mathbf{R})$ is such that $\theta(s) = 1$ if $|s| \leqslant 1/2$ and $\theta(s) = 0$ for $|s| \geqslant 1$.

Theorem 1. *Assume that for $u \in V(\Omega_N)$ and $a(x)|u|^{p+1} \in L^1(\Omega_N)$ the inequality*

$$\sum_{i,j=1}^{n} \int_{\Omega_N} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial (u\Theta_R)}{\partial x_i} \mathrm{d}x + \int_{\Omega_N} a(x)|u|^{p+1}\Theta_R \mathrm{d}x \leqslant - < f, u\Theta_R > \qquad (4)$$

holds for some $f \in V^*(\Omega_R)$ and any $R \leqslant N$. Then there exist a constant $C > 0$ independent of Ω, R and u such that if $R \in [1, N]$ we have

$$\int_{\Omega_{R/2}} |\nabla u|^2 \mathrm{d}x \leqslant R^{n-2-4/(p-1)} \left(C + R^{(4p-n(p-1))/(p-1)} \|f\|^2_{V^*(\Omega_R)} \right) \qquad (5)$$

and

$$\int_{\Omega_{R/2}} |u|^{p+1}\mathrm{d}x \leqslant R^{n-2(p+1)/(p-1)}\left(C + R^{(4p-n(p-1))/(p-1)}\|f\|^2_{V^*(\Omega_R)}\right). \tag{6}$$

Proof. Let $R = 1$. From (4) and the assumptions $p > 1$, $a(x) \geqslant a_0 = const > 0$ it follows that

$$\begin{aligned} &C_1 \int_{\Omega_1} |\nabla u|^2\theta^2(|x|)\mathrm{d}x + a_0 \int_{\Omega_1} |u|^{p+1}\theta^2(|x|)\mathrm{d}x \\ &\leqslant 2\epsilon \int_{\Omega_1} |\nabla u|^2\theta^2(|x|)\mathrm{d}x + C_2 \int_{\Omega_1} |u|^2|\theta'(|x|)|^2\mathrm{d}x \\ &\quad +|< f, u\theta^2 >|, \qquad C_1, C_2 = const. \end{aligned} \tag{7}$$

Using the Young inequality $\left(ab \leqslant \frac{a^s}{s} + \frac{b^q}{q}, \ \frac{1}{s} + \frac{1}{q} = 1\right)$, we obtain from (7) that

$$\begin{aligned} &\int_{\Omega_1} |\nabla u|^2\theta^2(|x|)\mathrm{d}x + a_0 \int_{\Omega_1} |u|^{p+1}\theta^2(|x|)\mathrm{d}x \\ &\leqslant C_3 \int_{\Omega_1} |u|^2\theta^K\theta^{-K}|\theta'|^2\mathrm{d}x + |< f, u\theta^2 >| \\ &\leqslant \delta \int_{\Omega_1} |u|^{p+1}\theta^2\mathrm{d}x + C_4 \int_{\Omega_1} |\theta'|^{2(p+1)/(p-1)}\theta^{-4/(p-1)}\mathrm{d}x \\ &\quad +|< f, u\theta^2 >|, \\ &\quad s = (p+1)/2, \ q = (p+1)/(p-1), K = 2s. \end{aligned} \tag{8}$$

By the definition of f

$$\begin{aligned} |< f, u\theta^2 >| &\leqslant C_5\|f\|^2_{V^*(\Omega_1)} + \delta_1\|u\theta^2\|^2_{H^1(\Omega_1)} \\ &\leqslant C_5\|f\|^2_{V^*(\Omega_1)} \\ &\quad +\delta_2\left(\int_{\Omega_1} u^2\theta^4\mathrm{d}x + \int_{\Omega_1} |\nabla u|^2\theta^2\mathrm{d}x + \int_{\Omega_1} u^2|\theta'|^2\mathrm{d}x\right). \end{aligned} \tag{9}$$

The last integral is already considered above (see (8)). Applying the Young inequality again we obtain

$$\int_{\Omega_1} u^2\theta^4 \mathrm{d}x \leqslant \delta_3 \int_{\Omega_1} u^{p+1}\theta^2 \mathrm{d}x + C_6 \int_{\Omega_1} \theta^{2+2\frac{(p+1)}{(p-1)}} \mathrm{d}x. \tag{10}$$

Here ϵ, δ, δ_1, δ_2 are arbitrarily small numbers. From (7)–(10) we have

$$\int_{\Omega_1} |\nabla u|^2\theta^2(|x|)\mathrm{d}x + \int_{\Omega_1} |u|^{p+1}\theta^2(|x|)\mathrm{d}x$$
$$\leqslant C_7\|f\|^2_{V^*(\Omega_1)} + C_8 \int_{\Omega_1} |\theta'|^{2(p+1)/(p-1)}\theta^{-4/(p-1)}\mathrm{d}x$$
$$+C_9 \int_{\Omega_1} \theta^{4p/(p-1)}\mathrm{d}x, \quad C_j = const. \tag{11}$$

Let us choose $\theta(s)$ such that $\theta'(s) = \mathcal{O}\left((1-s)^{m-1}\right)$ as $s \to 1$. If $m > (p+1)/(p-1)$, then the second integral is finite and we have

$$\int_{\Omega_1} |\nabla u|^2\theta^2 \mathrm{d}x + \int_{\Omega_1} |u|^{p+1}\theta^2 \mathrm{d}x \leqslant C_{10} + C_7\|f\|^2_{V^*(\Omega_1)}. \tag{12}$$

The last inequality yields us the inequality (5),(6) for $R = 1$. In order to get the inequality (5),(6) for $R > 1$ we introduce in (4) the change of variables

$$x' = \frac{x}{R}, \quad u(x) = R^{-2/(p-1)}v(Rx'). \tag{13}$$

In the new variables x', v we have (5),(6) for $R = 1$. If we write the inequalities thus obtained in terms of the variables x, u, we obtain (5),(6) for any $R \leqslant N$.

We use Theorem 1 to prove the existence and uniqueness results for the problem (1),(2).

Theorem 2. *Under the above conditions on the data of the problem (1),(2) there exists a unique weak solution $u(x)$ of problem (1),(2) such that $a(x)|u|^{p+1} \in L^1_{\mathrm{loc}}(\Omega)$ and the integral identity (3) holds also for $\varphi = u\theta^2\left(\frac{|x|}{R}\right)$ for any $R \geqslant 1$.*

Proof. (Existence) In order to construct a solution of the problem (1),(2) we consider the following boundary-value problem:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_N}{\partial x_j}\right) - a(x)|u_N|^{p-1}u_N = f(x) \text{ in } \Omega_N, \tag{14}$$

$$\frac{\partial u_N}{\partial \mu} = 0 \text{ on } \Gamma_N \cap B_N; \quad u_N = 0 \text{ on } (\Gamma_D \cap B_N) \cup \{x \in \Omega : |x| = N\}. \tag{15}$$

The existence and uniqueness of a solution $u_N \in H^1(\Omega_N)$ satisfying (14),(15) in the sense that $a(x)|u_N|^p \in L^1(\Omega_N)$ and the integral identity (3) holds with Ω replaced by Ω_N and for any $\varphi \in H^1(\Omega_N) \cap L^\infty(\Omega_N)$ with $\varphi = 0$ on $(\Gamma_D \cap B_N) \cup \{x \in \Omega : |x| = N\}$ is a consequence of the results of H. Brezis and F. Browder [5]. Their results also imply that $a(x)|u_N|^{p+1} \in L^1(\Omega_N)$ and that the integral identity also holds for $\varphi = u_N$. In fact, if $0 < R < N$ we have

$$\sum_{i,j=1}^{n} \int_{\Omega_R} a_{ij}\frac{\partial u_N}{\partial x_j}\frac{\partial(u_N\theta^2)}{\partial x_i}\mathrm{d}x + \int_{\Omega_R} a(x)|u_N|^{p+1}\theta^2\mathrm{d}x = \langle f, u_N\Theta^2\rangle \tag{16}$$

In order to prove (16) we take $\varphi = T_m(u_N)\theta^2$ in the integral identity associated to (3) but on Ω_N, where $T(s) = \min(m, |s|)\mathrm{sign}\ s$. Making $m \to \infty$ we obtain (16). By Theorem 1 we have

$$\int_{\Omega_{R/2}} \left(|\nabla u_N|^2 + |u_N|^{p+1}\right)\mathrm{d}x \leqslant C$$

for some constant C depending on R, but independent of N. By standard results we get that $\{u_N\}$ is bounded in $H^1(\Omega_R)$ and by diagonal extraction it follows that there exist u and a sub-sequence u_N such that $u_N \to u$ in $H^1_{\mathrm{loc}}(\Omega)$ weakly, in $L^{p+1}_{\mathrm{loc}}(\Omega)$ weakly, in $L^2_{\mathrm{loc}}(\Omega)$ strongly and almost everywhere in Ω. Passing to the limit as $N \to \infty$ in the integral identity for u_N we obtain (3).

(Uniqueness): Let us prove that the weak solution $u(x)$ of the problem (1),(2) constructed above is unique. Let u_1, u_2 be weak solutions of the problem (1),(2) such that $a(x)|u_i|^{p+1} \in L^1_{\mathrm{loc}}(\Omega)$, and the integral identity (3) holds for $\varphi = u_i\Theta_R$ for any $R > 1$, $i = 1, 2$. Let $v = u_1 - u_2$. Since $v \in V_{\mathrm{loc}}(\Omega) \cap L^{p+1}_{\mathrm{loc}}(\Omega)$, arguing as in the proof of (16), we can take $\varphi = v\Theta_R$ in the integral identity (3) associated to u_i. We get

$$\int_{\Omega_R} \sum_{i,j=1}^{n} a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial(v\Theta_R)}{\partial x_i}\mathrm{d}x + \int_{\Omega_R} A(x)|v|^{p+1}\Theta_R\mathrm{d}x = 0, \tag{17}$$

where

$$A(x) \equiv \begin{cases} \dfrac{a(x)\left[|u_1(x)|^{p-1}u_1(x) - |u_2(x)|^{p-1}u_2(x)\right]}{|u_1(x)-u_2(x)|^{p-1}(u_1(x)-u_2(x))}, & \text{if } u_1(x) \neq u_2(x), \\ a_0, & \text{if } u_1(x) = u_2(x). \end{cases}$$

It is easy to prove by considering the function

$$\frac{1-|y|^{p-1}y}{(1-y)|1-y|^{p-1}},$$

that $A(x) \geqslant A_0 = const > 0$. Therefore, Theorem 1 can be applied to the function v with $f \equiv 0$. According to Theorem 1 we have

$$\int_{\Omega_{R/2}} |\nabla v|^2 \mathrm{d}x \leqslant CR^{n-2-4/(p-1)}, \tag{18}$$

$$\int_{\Omega_{R/2}} |v|^{p+1} \mathrm{d}x \leqslant CR^{n-2(p+1)/(p-1)}, \qquad C = const. \tag{19}$$

If $1 < p < (n+2)/(n-2)$ or $n = 2$, then from (19) it follows that $v = 0$. If $p \geqslant (n+2)/(n-2)$, we shall prove that $v = 0$ in Ω by contradiction. Suppose that there exists $\alpha > 0$ such that $v(x) \geqslant \alpha$ on a set $\omega \subset \Omega$ and meas $\omega > 0$. It is easy to prove that the function $w = (v-\alpha)_+$, where $(v-\alpha)_+ = v - \alpha$ for $v - \alpha > 0$ and $(v-\alpha)_+ = 0$ for $v - \alpha \leqslant 0$, satisfies the equality

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial (w\Theta_R)}{\partial x_i} \mathrm{d}x + \int_\Omega A(x) \frac{(w+\alpha)^{p+1}}{w^{q+1}} w^{q+1} \Theta_R \mathrm{d}x = 0, \tag{20}$$

where $q < (n+2)/(n-2)$. Since $(w+\alpha)^{p+1}/w^{q+1} \geqslant b > 0$ for any $w > 0$ and $q < p$, where $b = const > 0$, we can apply Theorem 1 to w. We then have

$$\int_{\Omega_{R/2}} |w|^{q+1} \mathrm{d}x \leqslant CR^{n-2(q+1)/(q-1)}. \tag{21}$$

From (21) it follows that $w \equiv 0$ in Ω. That contradicts the assumption that $w > 0$ on $\omega \subset \Omega$ with meas $\omega > 0$.

Remark 1. The first result where the existence and uniqueness of a solution of the boundary-value problem (1),(2) with condition $\Gamma_N = \emptyset$ were proved without the growth conditions at infinity on f and u was by H.

Brezis [2]. The H. Brezis uniqueness result uses an explicit supersolution and his method cannot be applied to the case $\Gamma_N \neq \emptyset$.

Consider now the behavior as $|x| \to \infty$ of a weak solution of problem (1),(2). We study the case $f = 0$ in $\Omega \backslash B_{R'}$, where R' is some constant.

We consider the function $u \in H^1_{\text{loc}}(\Omega \backslash B_{R'})$ such that

$$a(x)|u|^{p+1} \in L^1_{\text{loc}}(\Omega \backslash B_{R'}),$$

$u \in L^\infty(\Omega \cap \partial B_{R'})$ and satisfies the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a(x)|u|^{p-1}u = f(x) \text{ in } D'(\Omega \backslash B_{R'}). \tag{22}$$

Theorem 3. *Assume that $u = 0$ on $\partial\Omega \backslash B_{R'}$, $a_{ij}(x) = const.$, $i,j = 1,\dots,n$. Then*

$$|u(x)| \leqslant C|x|^{-2/(p-1)} \text{ in } \Omega \backslash B_{R'}, \quad C = const > 0. \tag{23}$$

Proof. Let $N > R$. We set $\Omega_N^{R'} = \{x \in \Omega : R' < |x| < N\}$. Let v_N be a weak solution of the problem

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v_N}{\partial x_j}\right) - a(x)|v_N|^{p-1}v_N = 0 \text{ in } \Omega_N^{R'} \tag{24}$$

with the boundary conditions

$$v_N = u \text{ on } \overline{\Omega} \cap \partial B_{R'}, \text{ and } v_N = 0 \text{ on } (\partial\Omega \cap (B_N \backslash B_{R'}) \cup (\Omega \cap \partial B_N)). \tag{25}$$

Consider the function $V(x) = K|x|^{-2/(p-1)}$. It is easy to check that for $a_{ij}(x) = const.$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial V}{\partial x_j}\right) - a(x)|V|^{p-1}V \leqslant 0$$

and $V(x) \geqslant v_N(x)$ on $\Omega \cap \partial B_{R'}$, if $K > 0$ is a sufficiently large constant. Applying the maximum principle, we deduce that

$$|v_N(x)| \leqslant K|x|^{-2/(p-1)} \text{ in } x \in \Omega_N^{R'}. \tag{26}$$

Letting $N \to \infty$ we obtain (23), the assertion of the theorem.

The following result shows that the geometry of Ω can have influence on the decay of $u(x)$ as $|x| \to \infty$.

Theorem 4. *Assume that*

$$diameter(\Omega \cap \{x : |x| = N\}) \leqslant T, \quad \forall N \geqslant R', \tag{27}$$

where the constant T does not depend on N. Let u satisfy (22), $u = 0$ on $\partial\Omega \backslash B_{R'}$, $u \in H^1_{\text{loc}}(\Omega \backslash B_{R'})$. Then

$$|u(x)| \leqslant Ce^{-\alpha|x|}, \quad C, \alpha = const > 0. \tag{28}$$

Proof. Let $N > R'$ and let v_N be the solution of the problem (24),(25). It is easy to prove as in Theorem 3, using $V(x) = const.$, that $|v_N(x)| \leqslant C_1$, where $C_1 = const$ and C_1 does not depend on N. Taking $\varphi = v_N e^{\alpha|x|}\Psi^2(x)$ as a test function for (24), where $\Psi = 1$ for $|x| > R' + 1$, $\Psi = 0$ for $|x| < R' + \frac{1}{2}$, $0 \leqslant \Psi \leqslant 1$, $\Psi \in C^\infty(\mathbf{R}^n)$, we obtain

$$\begin{aligned} &C_1 \int\limits_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\Psi^2(x)\mathrm{d}x + a_0 \int\limits_{\Omega_N^{R'}} |v_N|^{p+1} e^{\alpha|x|}\Psi^2(x)\mathrm{d}x \\ &\leqslant \alpha C_2 \int\limits_{\Omega_N^{R'}} |v_N|^2 e^{\alpha|x|}\Psi^2 \mathrm{d}x + \alpha \int\limits_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\Psi^2 \mathrm{d}x \\ &+2 \int\limits_{\Omega_N^{R'+1/2} \backslash \Omega_N^{R'+1}} |\nabla v_N||v_N| e^{\alpha|x|}\Psi|\nabla\Psi|\mathrm{d}x. \end{aligned} \tag{29}$$

In order to estimate the first integral on the right-hand side of (29) we use the Friedrichs inequality

$$\int\limits_{\Omega_N^{R'}} |v_N|^2 e^{\alpha|x|}\mathrm{d}x \leqslant C_3 \int\limits_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\Psi^2 \mathrm{d}x, \tag{30}$$

where as a consequence of (27) the constant C_3 is independent of N. We have

$$\int\limits_{\Omega_N^{R'+1/2} \backslash \Omega_N^{R'+1}} |\nabla v_N||v_N| e^{\alpha|x|}\Psi|\nabla\Psi|\mathrm{d}x$$

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$$\leqslant \epsilon \int_{\Omega_N^{R'+1/2}\setminus\Omega_N^{R'+1}} |\nabla v_N|^2 e^{\alpha|x|}|\Psi|^2 \mathrm{d}x$$
$$+C_4 \int_{\Omega_N^{R'+1/2}\setminus\Omega_N^{R'+1}} |v_N|^2 e^{\alpha|x|}|\nabla\Psi|^2 \mathrm{d}x$$
$$\leqslant \epsilon \int_{\Omega_N^{R'+1/2}\setminus\Omega_N^{R'+1}} |\nabla v_N|^2 e^{\alpha|x|}|\Psi|^2 \mathrm{d}x$$
$$+C_5(\epsilon) \int_{\Omega_N^{R'+1/2}\setminus\Omega_N^{R'+1}} |v_N|^2 e^{\alpha|x|} \mathrm{d}x, \qquad (31)$$

where ϵ is an arbitrarily small positive number.

For sufficiently small α we have from (28),(29),(31) that

$$\int_{\Omega_N^{R'}} |\nabla v_N|^2 e^{\alpha|x|}\Psi^2 \mathrm{d}x + a_0 \int_{\Omega_N^{R'}} |v_N|^{p+1} e^{\alpha|x|}\Psi^2 \mathrm{d}x$$
$$\leqslant C_6 \int_{\Omega_N^{R'+1/2}\setminus\Omega_N^{R'+1}} |v_N|^2 e^{\alpha|x|} \mathrm{d}x. \qquad (32)$$

The last integral in (32) is bounded by a constant which does not depend on N. In order to prove it we cover $\Omega_N^{R'+1/2}\setminus\Omega_N^{R'+1}$ by balls of radius 1/2 and apply Theorem 1. From (32) we have

$$\int_{\Omega_N^{R'+1}} |\nabla v_N|^2 e^{\alpha|x|} \mathrm{d}x + \int_{\Omega_N^{R'+1}} |v_N|^{p+1} e^{\alpha|x|}\Psi^2 \mathrm{d}x \leqslant C_7, \qquad (33)$$

where the constant C_7 does not depend on N and $\alpha = const > 0$ is sufficiently small.

Since $v_N = 0$ on $\partial\Omega$ for $|x| > R'$, by the De Giorgi type theorem (see e.g. [7], Theorem 8.17) we have

$$\sup_{x\in B_{1/2}(x_0)\cap\Omega_N^{R'}} |v_N(x)|$$
$$\leqslant C_8 \left[\int_{B_1(x_0)\cap\Omega_N^{R'}} |v_N|^{p+1} \mathrm{d}x \right]^{\frac{1}{(1+p)}}$$

$$\leqslant C_9 \left[e^{-\alpha(|x_0|-1)} \int\limits_{B_1(x_0)\cap\Omega_N^{R'}} |v_N|^{p+1} e^{\alpha|x|} \mathrm{d}x \right]^{\frac{1}{(p+1)}}$$
$$\leqslant C_{10} \exp\{\alpha|x_0|\},$$

where $x_0 \in \Omega_N^{R'}$, $B_\rho(x_0) = \{x : |x - x_0| < \rho\}$, $\alpha = const > 0$ and C_{10} does not depend on N. Making $N \to \infty$ we obtain (28). This concludes the proof of the theorem.

The results of this section were obtained jointly with J.I. Diaz and published in a short communication [8].