The sign-solvability of a linear system implies that the signs of the entries of the solution (or at least some of the entries) are determined solely on the basis of the signs of the coefficients of the system. That it might be worthwhile and possible to investigate such linear systems was recognized by Samuelson in his classic book *Foundations of Economic Analysis*. Sign-solvability is part of a larger study which seeks to understand the special circumstances under which an algebraic, analytic, or geometric property of a matrix can be determined from the combinatorial arrangement of the positive, negative, and zero elements of the matrix. These are thus properties shared by all members of a qualitative class of matrices. Several classes of matrices arise in this way, notably sign-nonsingular matrices, L-matrices, S-matrices, and sign-stable matrices. The essential idea of a sign-nonsingular matrix arose in a different context in the key 1963 paper "Dimer statistics and phase transitions" by P. W. Kastelyn.

The large and diffuse body of literature connected with sign-solvability is presented as a coherent whole for the first time in this book, displaying it as a beautiful interplay between combinatorics (especially graph theory) and linear algebra. Results in the literature are presented in a new and organized way with many new connections established and with many new results and proofs. One of the features of this book is that algorithms that are implicit in many of the proofs have been explicitly described and their complexity has been commented on.

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Matrices of sign-solvable linear systems

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To Mona (so unexpected and so special)
(from RAB)

To Chanyoung
(from BLS)

Contents

Preface

The possibility of writing this book occurred to us in the late fall of 1991 when we were both participating in the program on Applied Linear Algebra at the Institute for Mathematics and its Applications (IMA) in Minnesota. A few years earlier we had been attracted to the subject of sign-solvability because of the beautiful interplay it afforded among linear algebra, combinatorics, and theoretical computer science (combinatorial algorithms). The subject, begun in 1947 by the economist P. Samuelson, was developed from various perspectives in the linear algebra, combinatorics, and economics literature. We thought that it would be a worthwhile project to organize the subject and to give a unified and self-contained presentation. Because there were no previous books or even survey papers on the subject, the tasks of deciding what was fundamental and how the material should be ordered for exposition had to be thought out very carefully. Our organization of the material has resulted in new connections among various results in the literature. In addition, many new results and many new and simpler proofs of previously established results are given throughout the book. We began the book in earnest in early 1992 and completed approximately three quarters of it while in residence at the IMA. After we returned to our home institutions, with the other duties that that entails, it was difficult to find the time for completing the book.

One of the features of this book is that we have explicitly described algorithms that are implicit in many of the proofs and have commented on their complexity. Throughout we have given credit for results that have previously occurred in the literature. There is a bibliography at the end of each chapter as well as a master bibliography (including some papers not cited in the text) at the end of the book.

That it might be worthwhile to investigate systems of linear equations for which the signs of the solution could be determined knowing only the signs

of its coefficients was recognized by Samuelson in his book *Foundations of Economic Analysis*. The mathematical study of sign-solvability, in particular of sign-nonsingular matrices, was begun by L. Bassett, J. Maybee and J. Quirk in their paper "Qualitative economics and the scope of the correspondence principle" in 1968. Since the appearance in 1984 of the paper "Signsolvability revisited" by V. Klee, R. Ladner, and R. Manber, there has been renewed interest in the subject. Indeed we were first attracted to sign-solvability and related topics by this paper. The essential idea of a sign-nonsingular matrix arose in a different context in the 1963 paper "Dimer statistics and phase transitions" by P.W. Kastelyn. A key paper in the development that proceeded from Kastelyn's work is the 1975 paper "A characterization of convertible (0, 1)-matrices" by C.H.C. Little. The connection between the two different points of view was made in RAB's 1988 paper "Counting permutations with restricted positions: Permanents of (0, 1)-matrices. A tale in four parts."

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