Part I

Vector Bundles on Algebraic Curves

Introduction

The following is the text of a course given at the Université Denis Diderot (Paris VII) in Spring 1991. Although it was aimed at students who had already followed a third level Algebraic Geometry course, in chapters 1 and 2, we review the fundamental statements and particularly those regarding the cohomology of coherent algebraic sheaves, the Serre Duality Theorem and the Riemann-Roch formula. This chapter contains only a few proofs and we refer the reader for more details to basic algebraic geometry texts such as the books of Hartshorne [31] and Shafarevich [57]. Properties connected with direct images and properties of flatness (the Semi-continuity Theorem and the criterion for flatness) are dealt with in chapter 4. In this chapter, we shall also describe the construction of the Hilbert-Grothendieck scheme of coherent sheaves with fixed Hilbert polynomial which are quotients of a given vector bundle over a projective algebraic curve.

The subject of this course is the problem of the classification of algebraic vector bundles on a connected, smooth, projective algebraic curve X of genus g. To simplify matters, we restrict ourselves to the case where the base field is $\mathbb{C}$, which allows us to view the degree as a topological invariant and to obtain the topological classification of such bundles immediately: in effect, they are classified by their rank and degree (cf. chapter 3). This allows us to study algebraic vector bundles with given rank r and degree d. This family is not bounded (cf. chapter 5) and so, to parametrize vector bundles by algebraic varieties, it is natural to introduce the notions of stability and semi-stability. Other bundles are classified by reducing them using a Harder-Narasimhan filtration.

One finds very quickly that, for large enough d, the set of isomorphism classes of semi-stable bundles of rank r and degree d can be identified with the set of orbits under a natural action of the group $\mathrm{SL}(H)$ on an invariant open subset Ω of the Hilbert-Grothendieck scheme $\mathbf{Hilb}^{r,d}(H\otimes \mathcal{O}_X)$ of coherent quotient modules of the trivial bundle $H \otimes \mathcal{O}_X$, where H is a vector space of dimension $\chi = d + r(1-g)$. The problem then is to equip the quotient with the structure of an algebraic variety. It is here that Mumford's geometric invariant theory arises (cf. [55], [47], [50]). This Hilbert scheme embeds as a closed sub-variety of projective space and the open subset Ω is actually the open set of semi-stable points under the $\mathrm{SL}(H)$ action. Mumford's theory shows that this open subset has a good quotient $\mathbf{M}(r,d)$ and that this good quotient is a projective algebraic variety, the points of which correspond to certain equivalence classes of semi-stable bundles: this is the notion of S-equivalence, which is defined in terms of Jordan-Hölder filtrations. For stable bundles, these equivalence classes reduce to isomorphism classes, which gives the set of isomorphism classes of stable vector bundles of fixed rank and degree the structure of a quasi-projective variety $\mathbf{M}^s(r,d)$.

The above technique, now very classical, goes back to the work of Mumford and Seshadri (cf. [55], [47], [51]). However, the embedding that we have chosen in order to regard the Hilbert scheme $\mathbf{Hilb}^{r,d}(H\otimes\mathcal{O}_X)$ as a closed sub-variety of a projective space differs from the one introduced by these authors: we make use of the polarization which Grothendieck introduced in its initial construction [29], which results in proofs which are very much easier and without doubt more natural (cf. chapter 7). It has the other great advantage that it can be generalized to singular curves without much change: and, moreover, the method lends itself to the construction of moduli spaces of semi-stable sheaves on varieties of larger dimension without major modification, as one can see by studying, for example, Simpson's article [58].

Chapter 8 is devoted to proving the theorems of existence, irreducibility and smoothness of the open subset of stable bundles. The smoothness criterion for the points of the Hilbert scheme $\mathbf{Hilb}^{r,d}(H, \mathcal{O}_X)$ is obtained following an idea of Grothendieck [29] and involves studying the possible extensions of the quotients parametrized by fat points, a method that goes by the name of the criterion of formal smoothness. The use of the criterion of formal smoothness here seems to be unavoidable because, in a neighbourhood of a point, the Hilbert scheme arises as an intersection of a finite family of determinantal varieties, so it is not easy to determine their codimension. We have no references for the theorems of chapter 8; the method used is the adaption of an exact sequence which J.-M. Drézet and the author introduced to study the existence of stable bundles on the projective plane [15] and which is explained more fully in Part II. We have included a problem at the end of Part I which is concerned with the study of algebraic vector bundles on elliptic curves and is obviously inspired by the article of M.F. Atiyah [1].

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Generalities

In this chapter, we shall review some of the basic facts about vector bundles. The main reference is the book of Shafarevich [57], Chapter VI. We shall assume that the reader has already met the notion of an algebraic sheaf or $\mathcal{O}_X$-module. We recall the main operations on vector bundles.

1.1 Definition

Let X be an algebraic variety[1]. A (*complex*) *linear fibration* over X is given by an algebraic variety E, a surjective morphism $p : E \to X$ of algebraic varieties and, for each point $x \in X$, a complex vector space structure on the fibre $p^{-1}(x)$. The variety E is called the *total space* of the fibration and the fibre over x is denoted by E_x. Given two fibrations $p : E \to X$ and $p' : E' \to X$, a morphism of varieties $f : E \to E'$ is a map of linear fibrations if it is compatible with the projections p and p', that is, $p' \circ f = p$, and if, for each $x \in X$, the induced map $f_x : E_x \to E'_x$ is linear. The bundle $X \times \mathbb{C}^r \to X$ given by projection to the first factor is called the *trivial fibration* of rank r. For each open set $U \subset X$, we write $E|_U$ for the fibration $p^{-1}(U) \to U$ given by restriction to U.

An *algebraic vector bundle* of rank r on X is a linear fibration $E \to X$ which is locally trivial in the following sense: for each point $x \in X$ there exist an open neighbourhood U of x and an isomorphism of fibrations

$$\varphi : E|_U \longrightarrow U \times \mathbb{C}^r.$$

[1] The base field is $\mathbb{C}$. In this chapter we will assume that the varieties are reduced but most of the results apply to the non-reduced case.

1.2 Transition functions

Such an isomorphism is called a *local chart* or a *trivialization* of the vector bundle which, abusing the notation, we denote by E. If $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^r$ and $\varphi_j : E|_{U_j} \to U_j \times \mathbb{C}^r$ are local charts over the open subsets U_i and U_j respectively then the change of chart defined over $U_{i,j} = U_i \cap U_j$ by

$$\varphi_i \circ \varphi_j^{-1} : U_{i,j} \times \mathbb{C}^r \longrightarrow U_{i,j} \times \mathbb{C}^r$$

takes the form $(x, v) \mapsto (x, g_{i,j}(x)v)$, where $g_{i,j} : U_{i,j} \to \mathrm{GL}(r, \mathbb{C})$ is a map of algebraic varieties. The maps $g_{i,j}$ are called *transition functions*. They satisfy the following conditions:

(a) $g_{i,j} = \mathrm{id}_{\mathbb{C}^r}$ over the open subset U_i,

(b) over the intersection $U_{i,j,k} = U_i \cap U_j \cap U_k$ we have $g_{i,k} = g_{i,j} g_{j,k}$.

1.3 The vector bundle associated to transition functions

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. For each $(i, j) \in I^2$, suppose that we are given a map $g_{i,j} : U_{i,j} \to \mathrm{GL}(r, \mathbb{C})$ satisfying conditions (a) and (b) above. We consider the quotient

$$E = \Big(\coprod_{i \in I} U_i \times \mathbb{C}^r\Big)/R$$

under the equivalence relation which identifies the points $(x, v) \in U_i \times \mathbb{C}^r$ with $(x', v') \in U_j \times \mathbb{C}^r$ when $x = x'$ and $v' = g_{i,j}(x)v$. This set is given the quotient topology. We then have a continuous projection $p : E \to X$ and a homeomorphism

$$E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^r$$

over U_i. We can then give the topological quotient space E the structure of an algebraic variety (separated and of finite type) induced from the algebraic variety structure of $U_i \times \mathbb{C}^r$. Over $U_{i,j}$, the structures induced from $U_i \times \mathbb{C}^r$ and from $U_j \times \mathbb{C}^r$ coincide. One can also carry over the vector space structure of the fibres: this vector space structure on E_x does not depend on i. Thus, we obtain a vector bundle $E \to X$ and by definition the above isomorphism gives a trivialization. So from a system of transition functions we have constructed a vector bundle over X.

1.4 Maps of vector bundles

Suppose that we are given two vector bundles E and F of ranks r and s, respectively, over X. Then a map from E to F is a map of the underlying linear fibrations. If $\varphi : E|_U \to U \times \mathbb{C}^r$ and $\psi : F|_U \to U \times \mathbb{C}^s$ are local charts for the bundles E and F over the same open subset U then the maps $\tilde{f} = \psi f \varphi^{-1} : U \times \mathbb{C}^r \to U \times \mathbb{C}^r$ are the local expressions of f in the charts φ and ψ, and are written as

$$(x, v) \mapsto (x, g(x)v),$$

where $g : U \to L(\mathbb{C}^r, \mathbb{C}^s)$ is a map of algebraic varieties from U to the vector space $L(\mathbb{C}^r, \mathbb{C}^s)$ of linear maps $\mathbb{C}^r \to \mathbb{C}^s$.

Proposition 1.4.1 *Let $f : E \to F$ be a map of algebraic vector bundles and let x_0 be a point such that f_{x_0} is invertible. Then there exists an open neighbourhood U of x_0 such that $f|_U : E|_U \to F|_U$ is an isomorphism.*

Proof It suffices to prove this for the local expression for f. The hypotheses imply that $g(x_0)$ is invertible. Since g is continuous, $g(x)$ is invertible on an open neighbourhood U of x_0 and we can use this to construct the inverse of f. □

1.5 Associated bundles

Given two bundles E and F over X, the *direct sum* of E and F is defined, as a set, by

$$E \oplus F = \coprod_{x \in X} (E_x \oplus F_x).$$

To equip this set with the structure of an algebraic variety we start by choosing local charts $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^r$ and $\psi_i : F|_{U_i} \to U_i \times \mathbb{C}^s$ over the same open set U_i. Consider the bijections

$$(E \oplus F)|_{U_i} \to U_i \times (\mathbb{C}^r \oplus \mathbb{C}^s)$$

associated to φ_i and ψ_i and carry over the algebraic structure from the right hand side using this bijection. To see that this structure is independent of the choice of charts, it suffices to calculate the change of charts for $E \oplus F$: over the open set $U_{i,j}$ it is given by the transition functions $U_{i,j} \to \mathrm{GL}(\mathbb{C}^r \oplus \mathbb{C}^s)$ defined by the matrix

$$\begin{pmatrix} g_{i,j} & 0 \\ 0 & h_{i,j} \end{pmatrix},$$

where $g_{i,j}$ and $h_{i,j}$ are the transition functions of E and F respectively.

In the same way one can construct the bundles $\mathcal{H}om(E, F)$, $E \otimes F$, the dual bundle E^*, the exterior power $\Lambda^k(E)$ and the symmetric power $S^k(E)$.

1.6 Sub-bundles

Let $p : E \to X$ be an algebraic vector bundle of rank r over X. A *sub-bundle of rank* m of E is a sub-variety $F \subset E$ such that, for each $x \in X$, the intersection $F \cap E_x$ is a vector sub-space of E_x of dimension m and such that the induced fibration

$$p|_F : F \longrightarrow X$$

is locally trivial.

Example.
Let $X = \mathbf{P}_n(\mathbb{C})$ be n-dimensional projective space. We let $(x_0, \ldots, x_n)$ denote the homogeneous coordinates of a point $x \in X$. Consider the set of pairs $(x, v) \in X \times \mathbb{C}^{n+1}$ such that $v \in x$. This is a closed sub-variety H of $X \times \mathbb{C}^{n+1}$. Over the open set U of X defined by $x_i \neq 0$, W is defined by the equations

$$v_j = \frac{x_j}{x_i} v_i, \quad j = 0, \ldots, n.$$

The fibre H_x of $H \to X$ over a point x can be identified with the line x in $\mathbb{C}^{n+1}$. Then we have an isomorphism of fibrations

$$U_i \times \mathbb{C} \longrightarrow H|_{U_i}$$

given by $(x, t) \mapsto (x, v)$, where v is defined by $v_j = \frac{x_j}{x_i} t$ for $j = 0, \ldots, n$. It follows that H is a rank 1 vector bundle over X. This bundle is called the *Hopf bundle* and its dual is denoted $\mathcal{O}(1)$.

1.7 Quotient bundles

Let $F \subset E$ be a sub-bundle of an algebraic vector bundle E. Consider the family of vector spaces over X

$$E/F = \coprod_{x \in X} (E_x/F_x).$$

Any map of vector bundles $f : E \to G$ which vanishes on F factorizes uniquely through a map $\bar{f} : E/F \to G$.

Proposition 1.7.1 *There exists a unique vector bundle structure on $E/F \to X$ which satisfies the following universal property: for each map $f : E \to G$ which vanishes on F, the map $\bar{f}$ is a map of algebraic vector bundles.*

Proof Let x be a point of X. Over an open neighbourhood U of x we can find a vector sub-bundle S of $E|_U$ such that $F_x \oplus S_x = E_x$. Then the canonical map of vector bundles

$$F|_U \oplus S \longrightarrow E|U$$

is an isomorphism over the point x and so gives rise to an isomorphism over some open neighbourhood V of x by Proposition 1.4.1. This gives rise to a bijection $k : S|_V \to (E/F)|_V$ which is compatible with the vector space structures on the fibres. This identification gives rise to a vector bundle structure on $(E/F)|_V$. It is clear that the structure we obtain in this way over V is independent of the choice of S and of V, the induced structures match up. So we obtain a vector bundle structure on E/F which satisfies the given universal property because $\bar{f} \circ k$ is the restriction of f to $S|_V$ and, therefore, a map of vector bundles. □

Proposition 1.7.2 *Let $f : E \to F$ be a map of vector bundles. Suppose that the rank of f_x remains constant as x varies over X. Then* $\ker f$ *and* $\operatorname{Im} f$ *are sub-bundles of E and F respectively.*

Proof The question is local on X. Let x be a point of X. Choose sub-bundles K and S of $E|_V$ and L and T of $F|_V$ over a suitable open neighbourhood V of x so that $K_x = \ker f_x$, $L_x = \operatorname{Im} f_x$ and such that

$$K \oplus S = E|_V \quad \text{and} \quad L \oplus T = F|_V.$$

With respect to these direct sums, we write the map f as

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Over the point x, the maps a, c and d are zero and b is invertible. By Proposition 1.4.1, b is an isomorphism over a neighbourhood of x. Then the kernel of f is given by the pairs $(u, v) \in K \oplus S$ such that $(c - db^{-1}a)u = 0$ and $v = -b^{-1}au$. It now follows that the map $c - db^{-1}a$ has constant rank which must be zero because it vanishes at x. Then the isomorphism $(u, v) \mapsto (u, v + b^{-1}au)$ maps the kernel of f to the first factor of the direct sum and so it is a sub-bundle. Similarly, the image of f is identified with the set of pairs $(\xi, \eta) \in L \oplus T$ such that $\eta = db^{-1}\xi$ and the isomorphism $(\xi, \eta) \mapsto (\xi, \eta - db^{-1}\xi)$ maps the image of f onto the first factor of the direct sum. Consequently, this image is also a sub-bundle[1]. □

Corollary 1.7.3 *Let $f : E \to F$ be a map of vector bundles. If f is surjective then* $\ker f$ *is a sub-bundle of E. If f is injective then* $\operatorname{Im} f$ *is a vector sub-bundle of F.*

Example.
Let X be a smooth algebraic sub-variety of $\mathbb{C}^n$ of dimension m defined by an equation $f = 0$, where $f : \mathbb{C}^m \to \mathbb{C}^k$ is a polynomial map. Then the differential $(x, v) \mapsto (x, d_x f(v))$ has constant rank over X. Its kernel is a sub-bundle of rank n of $X \times \mathbb{C}^m$ called the *tangent bundle* to X and denoted $T(X)$. If X' is another smooth sub-variety of $\mathbb{C}^{m'}$ then each map $\varphi : X \to X'$ induces a map of vector bundles $T\varphi : T(X) \to T(X')$.

Example.
More generally, let X be a smooth algebraic variety of dimension n. We can always find a cover of X by open affine sets U_i and for each i an isomorphism $\varphi_i : U_i \to U_i'$ to a closed sub-variety of $\mathbb{C}^{n_i}$. The *tangent space* at x to X is given by $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring of X at the point x. Consider the space

$$T(X) = \coprod_{x \in X} T_x X \longrightarrow X.$$

The isomorphism φ_i induces an isomorphism $T_x\varphi_i : T_x \to T_{\varphi_i(x)}(U_i')$. So we have a bijection

$$T(X)|_{U_i} \longrightarrow T(U_i')$$

defined by $(x, v) \mapsto (\varphi_i(x), T_x\varphi_i(v))$ which is linear in each fibre and allows us to transfer the vector bundle structure to $T(X)|_{U_i}$. Over the open set $U_{i,j} = U_i \cap U_j$, such structures defined by φ_i and φ_j coincide.

1.8 Sections

Let $p : E \to X$ be a vector bundle of rank r over an algebraic variety X. We define a *regular section* of E over an open set U to be a map $s : U \to E$ of algebraic varieties such that $p(s(x)) = x$ for all $x \in U$.

[1] The statement of Proposition 1.7.2 does not remain true for a non-reduced variety but the corollary is still true.