

Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)

CAMBRIDGE STUDIES IN  
ADVANCED MATHEMATICS : 47

EDITORIAL BOARD

D.J.H. GARLING, T. TOM DIECK, P. WALTERS

AN ALGEBRAIC INTRODUCTION TO  
COMPLEX PROJECTIVE GEOMETRY. 1

Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)*Already published*

- 1 W.M.L. Holcombe *Algebraic automata theory*
- 2 K. Petersen *Ergodic theory*
- 3 P.T. Johnstone *Stone spaces*
- 4 W.H. Schikhof *Ultrametric calculus*
- 5 J.-P. Kahane *Some random series of functions, 2nd edition*
- 6 H. Cohn *Introduction to the construction of class fields*
- 7 J. Lambek & P.J. Scott *Introduction to higher-order categorical logic*
- 8 H. Matsumura *Commutative ring theory*
- 9 C.B. Thomas *Characteristic classes and the cohomology of finite groups*
- 10 M. Aschbacher *Finite group theory*
- 11 J.L. Alperin *Local representation theory*
- 12 P. Koosis *The logarithmic integral I*
- 13 A. Pietsch *Eigenvalues and s-numbers*
- 14 S.J. Patterson *An introduction to the theory of the Riemann zeta-function*
- 15 H.J. Baues *Algebraic homotopy*
- 16 V.S. Varadarajan *Introduction to harmonic analysis on semisimple Lie groups*
- 17 W. Dicks & M. Dunwoody *Groups acting on graphs*
- 18 L.J. Corwin & F.P. Greenleaf *Representations of nilpotent Lie groups and their applications*
- 19 R. Fritsch & R. Piccinini *Cellular structures in topology*
- 20 H. Klingen *Introductory lectures on Siegel modular forms*
- 21 P. Koosis *The logarithmic integral II*
- 22 M.J. Collins *Representations and characters of finite groups*
- 24 H. Kunita *Stochastic flows and stochastic differential equations*
- 25 P. Wojtaszczyk *Banach spaces for analysts*
- 26 J.E. Gilbert & M.A.M. Murray *Clifford algebras and Dirac operators in harmonic analysis*
- 27 A. Frohlich & M.J. Taylor *Algebraic number theory*
- 28 K. Goebel & W.A. Kirk *Topics in metric fixed point theory*
- 29 J.F. Humphreys *Reflection groups and Coxeter groups*
- 30 D.J. Benson *Representations and cohomology I*
- 31 D.J. Benson *Representations and cohomology II*
- 32 C. Allday & V. Puppe *Cohomological methods in transformation groups*
- 33 C. Soulé et al *Lectures on Arakelov geometry*
- 34 A. Ambrosetti & G. Prodi *A primer of nonlinear analysis*
- 35 J. Palis & F. Takens *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*
- 36 M. Auslander, I. Reiten & S. Smalø *Representation theory of Artin algebras*
- 37 Y. Meyer *Wavelets and operators*
- 38 C. Weibel *An introduction to homological algebra*
- 39 W. Bruns & J. Herzog *Cohen-Macaulay rings*
- 40 V. Snaith *Explicit Brauer induction*
- 41 G. Laumon *Cohomology of Drinfeld modular varieties I*
- 42 E.B. Davies *Spectral theory and differential operators*
- 43 J. Diestel, H. Jarchow & A. Tonge *Absolutely summing operators*
- 44 P. Mattila *Geometry of sets and measures in Euclidean spaces*
- 45 R. Pinsky *Positive harmonic functions and diffusion*
- 46 G. Tenenbaum *Introduction to analytic and probabilistic number theory*
- 50 I. Porteous *Clifford algebras and the classical groups*

Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)

# An Algebraic Introduction to Complex Projective Geometry

## 1. Commutative algebra

Christian Peskine

*Professor at University Paris VI, Pierre et Marie Curie*



Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521480727](http://www.cambridge.org/9780521480727)

© Cambridge University Press 1996

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 1996

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-48072-7 hardback

Transferred to digital printing 2007

# Contents

<b>1</b>	<b>Rings, homomorphisms, ideals</b>	<b>1</b>
1.1	Ideals. Quotient rings . . . . .	2
1.2	Operations on ideals . . . . .	6
1.3	Prime ideals and maximal ideals . . . . .	7
1.4	Nilradicals and Jacobson radicals . . . . .	10
1.5	Comaximal ideals . . . . .	11
1.6	Unique factorization domains (UFDs) . . . . .	12
1.7	Exercises . . . . .	14
<b>2</b>	<b>Modules</b>	<b>17</b>
2.1	Submodules. Homomorphisms. Quotient modules . . . . .	18
2.2	Products and direct sums . . . . .	20
2.3	Operations on the submodules of a module . . . . .	21
2.4	Free modules . . . . .	22
2.5	Homomorphism modules . . . . .	24
2.6	Finitely generated modules . . . . .	25
2.7	Exercises . . . . .	28
<b>3</b>	<b>Noetherian rings and modules</b>	<b>29</b>
3.1	Noetherian rings . . . . .	29
3.2	Noetherian UFDs . . . . .	31
3.3	Primary decomposition in Noetherian rings . . . . .	32
3.4	Radical of an ideal in a Noetherian ring . . . . .	33
3.5	Back to primary decomposition in Noetherian rings . . . . .	34
3.6	Minimal prime ideals . . . . .	35
3.7	Noetherian modules . . . . .	36
3.8	Exercises . . . . .	37
<b>4</b>	<b>Artinian rings and modules</b>	<b>39</b>
4.1	Artinian rings . . . . .	39
4.2	Artinian modules . . . . .	43
4.3	Exercises . . . . .	43

<b>5</b>	<b>Finitely generated modules over Noetherian rings</b>	<b>45</b>
5.1	Associated prime ideals . . . . .	45
5.2	Finite length modules . . . . .	48
5.3	Finitely generated modules over principal ideal rings . . . . .	51
5.4	The Artin–Rees lemma and Krull’s theorem . . . . .	55
5.5	Exercises . . . . .	56
<b>6</b>	<b>A first contact with homological algebra</b>	<b>59</b>
6.1	Some abelian categories . . . . .	59
6.2	Exact sequences . . . . .	61
6.3	Tensor products and homomorphism modules . . . . .	65
6.4	Dualizing module on an artinian ring . . . . .	68
6.5	Gorenstein artinian rings . . . . .	73
6.6	Exercises . . . . .	75
<b>7</b>	<b>Fractions</b>	<b>79</b>
7.1	Rings of fractions . . . . .	79
7.2	Fraction modules . . . . .	82
7.3	Support of a module . . . . .	86
7.4	Localization of ideals . . . . .	92
7.5	Localization and UFDs . . . . .	94
7.6	Localization and primary decomposition . . . . .	96
7.7	Back to minimal prime ideals . . . . .	98
7.8	Localization and associated prime ideals . . . . .	99
7.9	Exercises . . . . .	101
<b>8</b>	<b>Integral extensions of rings</b>	<b>103</b>
8.1	Algebraic elements, integral elements . . . . .	103
8.2	Finite extensions, integral extensions . . . . .	105
8.3	Going-up and going-down theorems . . . . .	108
8.4	Exercises . . . . .	112
<b>9</b>	<b>Algebraic extensions of fields</b>	<b>113</b>
9.1	Finite extensions . . . . .	113
9.2	$K$ -isomorphisms in characteristic zero . . . . .	116
9.3	Normal extensions . . . . .	117
9.4	Trace and norm . . . . .	121
9.5	Roots of one and cyclic Galois groups . . . . .	123
9.6	Exercises . . . . .	126
<b>10</b>	<b>Noether’s normalization lemma</b>	<b>129</b>
10.1	Transcendence degree . . . . .	129
10.2	The normalization lemma . . . . .	130
10.3	Hilbert’s Nullstellensatz . . . . .	133

*Contents*

vii

10.4	Jacobson rings . . . . .	136
10.5	Chains of prime ideals in geometric rings . . . . .	136
10.6	Height and dimension . . . . .	138
10.7	Dimension of geometric rings . . . . .	141
10.8	Exercises . . . . .	143
<b>11</b>	<b>Affine schemes</b>	<b>145</b>
11.1	The affine space $\mathbb{A}_n$ . . . . .	145
11.2	Affine schemes . . . . .	147
11.3	Closed and open subschemes of an affine scheme . . . . .	149
11.4	Functions defined on an open set . . . . .	153
11.5	Dimension of an affine scheme . . . . .	155
11.6	Irreducible components of an affine scheme . . . . .	156
11.7	Exercises . . . . .	157
<b>12</b>	<b>Morphisms of affine schemes</b>	<b>159</b>
12.1	Morphisms of affine schemes . . . . .	159
12.2	Immersions of affine schemes . . . . .	161
12.3	Local description of a morphism . . . . .	164
12.4	Product of affine schemes . . . . .	165
12.5	Dimension, product and intersection . . . . .	167
12.6	Dimension and fibres . . . . .	169
12.7	Finite morphisms . . . . .	170
12.8	Exercises . . . . .	171
<b>13</b>	<b>Zariski's main theorem</b>	<b>173</b>
13.1	Proof of Zariski's main theorem . . . . .	174
13.2	A factorization theorem . . . . .	178
13.3	Chevalley's semi-continuity theorem . . . . .	178
13.4	Exercises . . . . .	181
<b>14</b>	<b>Integrally closed Noetherian rings</b>	<b>183</b>
14.1	Reduced Noetherian rings . . . . .	183
14.2	Integrally closed Noetherian rings . . . . .	185
14.3	Discrete valuation rings. Dedekind rings . . . . .	187
14.4	Integral extensions of Noetherian domains . . . . .	190
14.5	Galois group and prime ideals . . . . .	192
14.6	Exercises . . . . .	194
<b>15</b>	<b>Weil divisors</b>	<b>197</b>
15.1	Weil divisors . . . . .	197
15.2	Reflexive rank-one modules and Weil divisors . . . . .	201
15.3	Exercises . . . . .	209

Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)

viii

*Contents*

<b>16 Cartier divisors</b>	<b>211</b>
16.1 Cartier divisors . . . . .	211
16.2 The Picard group . . . . .	217
16.3 Exercises . . . . .	220
<i>Subject index</i>	<i>225</i>
<i>Symbols index</i>	<i>229</i>



Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)

# Introduction

Commutative algebra is the theory of commutative rings and their modules. Although it is an interesting theory in itself, it is generally seen as a tool for geometry and number theory. This is my point of view. In this book I try to organize and present a cohesive set of methods in commutative algebra, for use in geometry. As indicated in the title, I maintain throughout the text a view towards complex projective geometry.

In many recent algebraic geometry books, commutative algebra is often treated as a poor relation. One occasionally refers to it, but only reluctantly. It also suffers from having attracted too much attention thirty years ago. One or several texts are usually recommended: the “Introduction to Commutative Algebra” by Atiyah and Macdonald is a classic for beginners and Matsumura’s “Commutative rings” is better adapted for more advanced students. Both these books are excellent and most readers think that there is no need for any other. Today’s students seldom consult Bourbaki’s books on commutative algebra or the algebra part in the E.G.A. of Grothendieck and Dieudonné.

With this book, I want to prepare systematically the ground for an algebraic introduction to complex projective geometry. It is intended to be read by undergraduate students who have had a course in linear and multilinear algebra and know a bit about groups. They may have heard about commutative rings before, but apart from  $\mathbb{Z}$  and polynomial rings in one variable with coefficients in  $\mathbb{R}$  or  $\mathbb{C}$ , they have essentially worked with fields. I had to develop quite a lot of language new to them, but I have been careful to articulate all chapters around at least one important theorem. Furthermore I have tried to stimulate readers, whenever their attention may be drifting away, by presenting an example, or by giving them an exercise to solve.

In the first eight chapters, the general theory of rings and modules is developed. I put as much emphasis on modules as on rings; in modern algebraic geometry, sheaves and bundles play as important a role as varieties. I had to decide on the amount of homological algebra that should be included and on the form it should take. This is difficult since the border between commutative and homological algebras is not well-defined. I made several conventional choices. For example, I did not elaborate immediately on the homological

Cambridge University Press

978-0-521-48072-7 - An Algebraic Introduction to Complex Projective Geometry: Commutative Algebra

Christian Peskine

Frontmatter

[More information](#)

nature of length. But quite early on, when studying dualizing modules on Artinian rings in chapter six, I used non-elementary homological methods. In chapter seven, I have been particularly careful on rings and modules of fractions, hoping to prepare readers for working with sheaves.

I wanted this book to be self-contained. Consequently, the basic Galois theory had to be included. I slipped it in at the end of this first part, in chapter nine, just after the study of integral ring extensions.

Now, our favourite ring is  $\mathbb{C}[X_1, \dots, X_n]$ , the polynomial ring in several variables with coefficients in the field of complex numbers. We can derive many rings from this one by natural algebraic procedures, even though our purposes are geometric. Quotient rings of  $\mathbb{C}[X_1, \dots, X_n]$ , in other words, finitely generated  $\mathbb{C}$ -algebras, and their fraction rings are the basic objects from chapter ten to chapter thirteen. Noether's normalisation lemma and Hilbert's Nullstellensatz, two splendid theorems of commutative algebra, concern these rings and are at the heart of algebraic geometry. With these results in view, I discuss the notion of dimension and move heartily towards geometry by introducing affine complex schemes and their morphisms. I can then present and prove two other important geometric results, a local version of Zariski's main theorem and Chevalley's semi-continuity theorem.

From chapter fourteen on, I have tried to provide a solid background for modern intersection theory by presenting a detailed study of Weil and Cartier divisors.

In order to keep this book short, I have had to make many painful choices. Several of the chapters that I have deleted from this text will appear in a second book, intended for graduate students, and devoted to homological algebra and complex projective geometry.

I have been careless with the historical background. But I have been careful in developing the material slowly, at least initially, though it does become progressively more difficult as the text proceeds. When necessary, examples and exercises are included within chapters. I allow myself to refer to some of those. All chapters are followed by a series of exercises. Many are easy and a few are more intricate; readers will have to make their own evaluation!

I would like to thank the many students, undergraduates or graduates, whom I taught, or with whom I discussed algebra, at the universities of Strasbourg, Oslo and Paris VI. I have tried to attract each of them to algebra and algebraic geometry. Special thanks are due to Benedicte Basili who wrote a first set of notes from my graduate course in algebraic geometry and introduced me to LaTeX.

My wife Vivi has been tremendously patient while I was writing this first book. I hope very much that some readers will think that she did well in being so.