GEOMETRIC ALGEBRA FOR PHYSICISTS

Geometric algebra is a powerful mathematical language with applications across a range of subjects in physics and engineering. Written by two of the leading researchers in the field, this book is a complete guide to the current state of the subject.

Early chapters provide a self-contained development of geometric algebra and form the basis of an undergraduate lecture course. Topics covered include new techniques for handling rotations in arbitrary dimensions, and the links between rotations, bivectors and the structure of the Lie groups. Following chapters extend the concept of a complex analytic function theory to arbitrary dimensions. This has applications in quantum theory and electromagnetism. All four Maxwell equations are united into one single equation, and new techniques are discussed for its solution. Later chapters cover some advanced topics in physics, including non-Euclidean geometry, quantum entanglement and gauge theories. The final chapters describe the construction of a gauge theory of gravitation in Minkowski spacetime. Using the tools of geometric algebra, advanced applications such as black holes and cosmic strings are explored.

This book will be of interest to researchers working in the fields of geometry, relativity and quantum theory. It can also be used as a textbook for advanced undergraduate and graduate courses on the physical applications of geometric algebra.

CHRIS DORAN obtained his PhD from the University of Cambridge, having gained a distinction in Part III of his undergraduate degree. He was elected a Junior Research Fellow of Churchill College, Cambridge in 1993, was made a Lloyd's of London Fellow in 1996 and was the Schlumberger Interdisciplinary Research Fellow of Darwin College, Cambridge between 1997 and 2000. He is currently a Fellow of Sidney Sussex College, Cambridge and holds an EPSRC Advanced Fellowship. Dr Doran has published widely on aspects of mathematical physics and is currently researching applications of geometric algebra in engineering and computer science.

ANTHONY LASENBY is Professor of Astrophysics and Cosmology at the University of Cambridge, and is currently Head of the Astrophysics Group and the Mullard Radio Astronomy Observatory in the Cavendish Laboratory. He began his astronomical career with a PhD at Jodrell Bank, specialising in the Cosmic Microwave Background, which has been a major subject of his research ever since. After a brief period at the National Radio Astronomy Observatory in America, he moved from Manchester to Cambridge in 1984, and has been at the Cavendish since then. He is the author or coauthor of nearly 200 papers spanning a wide range of fields, from early universe cosmology to computer vision. His introduction to geometric algebra came in 1988, when he encountered the work of David Hestenes for the first time, and since then he has been developing geometric algebra techniques and employing them in his research in many areas.

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CHRIS DORAN
and
ANTHONY LASENBY
University of Cambridge

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Contents

Preface

The ideas and concepts of physics are best expressed in the language of mathematics. But this language is far from unique. Many different algebraic systems exist and are in use today, all with their own advantages and disadvantages. In this book we describe what we believe to be the most powerful available mathematical system developed to date. This is *geometric algebra*, which is presented as a new mathematical tool to add to your existing set as either a theoretician or experimentalist. Our aim is to introduce the new techniques via their applications, rather than as purely formal mathematics. These applications are diverse, and throughout we emphasise the unity of the mathematics underpinning each of these topics.

The history of geometric algebra is one of the more unusual tales in the development of mathematical physics. William Kingdon Clifford introduced his geometric algebra in the 1870s, building on the earlier work of Hamilton and Grassmann. It is clear from his writing that Clifford intended his algebra to describe the geometric properties of vectors, planes and higher-dimensional objects. But most physicists first encounter the algebra in the guise of the Pauli and Dirac matrix algebras of quantum theory. Few then contemplate using these unwieldy matrices for practical geometric computing. Indeed, some physicists come away from a study of Dirac theory with the view that Clifford's algebra is inherently quantum-mechanical. In this book we aim to dispel this belief by giving a straightforward introduction to this new and fundamentally different approach to vectors and vector multiplication. In this language much of the standard subject matter taught to physicists can be formulated in an elegant and highly condensed fashion. And the portability of the techniques we discuss enables us to reach a range of advanced topics with little extra work.

This book is intended to be of interest to both students and researchers in physics. The early chapters grew out of an undergraduate lecture course that we have run for a number of years in the Physics Department at Cambridge University. We are indebted to the students who attended the early versions of this course, and helped to shape the material into a form suitable for undergraduate tuition. These early chapters require little more than a basic knowledge of linear algebra and vector geometry, and some exposure to classical mechanics. More advanced physical concepts are introduced as the book progresses.

A number of themes run throughout this book. The first is that geometric algebra enables us to express fundamental physics in a language that is free from coordinates or indices. Coordinates are only introduced later, when the geometry of a given problem is clear. This approach gives many equations a degree of clarity which is lost in tensor algebra. A second theme is the way in which rotations are handled in geometric algebra through the use of *rotors*. This approach extends to arbitrary spaces the idea of using a complex phase to rotate in a plane. Rotor techniques can be applied in spaces of arbitrary signature and are particularly well suited to formulating Lorentz and conformal transformations. The latter are central to our treatment of non-Euclidean geometry. Rotors also provide a framework for studying Lie groups and Lie algebras, and are essential to our discussion of gauge theories.

The third theme is the invertibility of the geometric product of vectors, which makes it possible to divide by a vector. This idea extends to the vector derivative, which has an inverse in the form a first-order Green's function. The vector derivative and its inverse enable us to extend complex analytic function theory to arbitrary dimensions. This theory is perfectly suited to electromagnetism, as all four Maxwell equations can be combined into a single spacetime equation involving the invertible vector derivative. The same vector derivative appears in the Dirac theory, and is central to the gauge treatment of gravitation which dominates the final two chapters of this book.

This book would not have been possible without the help and encouragement of a large number of people. We thank Stephen Gull for helping initiate much of the research described here, for his constant advice and criticism, and for use of a number of his figures. We also thank David Hestenes for all his work in shaping the modern subject of geometric algebra and for his constant encouragement. Special mention must be made of our many collaborators, in particular Joan Lasenby, Anthony Challinor, Leo Dorst, Tim Havel, Antony Lewis, Mark Ashdown, Frank Sommen, Shyamal Somaroo, Jeff Tomasi, Bill Fitzgerald, Youri Dabrowski and Mike Hobson. Special thanks also goes to Mike for his help with Latex and explaining the intricacies of the CUP style files. We thank the Physics Department of Cambridge University for the use of their facilities, and for the range of technical advice and expertise we regularly called on. Finally we thank everyone at Cambridge University Press who helped in the production of this book.

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not complaining about the lost evenings as I worked on this book. I promise to finish the decorating now it is complete.

AL thanks Joan and his children Robert and Alison for their constant enthusiasm and support, and their patience in the face of many explanations of topics from this book.

Cambridge — *C.J.L. Doran*
July 2002 — *A.N. Lasenby*

Notation

The subject of vector geometry in general, and geometric algebra in particular, suffers from a profusion of notations and conventions. In short, there is no single convention that is perfectly suited to the entire range of applications of geometric algebra. For example, many of the formulae and results given in this book involve arbitrary numbers of vectors and are valid in vector spaces of arbitrary dimensions. These formulae invariably look neater if one does not embolden all of the vectors in the expression. For this reason we typically choose to write vectors in a lower case italic script, a, and more general multivectors in upper case italic script, M. But in some applications, particularly mechanics and dynamics, one often needs to reserve lower case italic symbols for coordinates and scalars, and in these situations writing vectors in bold face is helpful. This convention in adopted in chapter 3.

For many applications it is useful to have a notation which distinguishes frame vectors from general vectors. In these cases we write the former in an upright font as $\{\mathsf{e}_i\}$. But this notation looks clumsy in certain settings, and is not followed rigourously in some of the later chapters. In this book our policy is to ensure that we adopt a consistent notation within each chapter, and any new or distinct features are explained either at the start of the chapter or at their point of introduction.

Some conventions are universally adopted throughout this book, and for convenience we have gathered together a number of these here.

(i) The geometric (or Clifford) algebra generated by the vector space of signature (p, q) is denoted $\mathcal{G}(p, q)$. In the first three chapters we employ the abbreviations $\mathcal{G}_2$ and $\mathcal{G}_3$ for the Euclidean algebras $\mathcal{G}(2, 0)$ and $\mathcal{G}(3, 0)$. In chapter 4 we use $\mathcal{G}_n$ to denote all algebras $\mathcal{G}(p, q)$ of total dimension n.

(ii) The geometric product of A and B is denoted by juxtaposition, AB.

(iii) The inner product is written with a centred dot, $A \cdot B$. The inner product is only employed between homogeneous multivectors.

(iv) The outer (exterior) product is written with a wedge, $A \wedge B$. The outer product is also only employed between homogeneous multivectors.

(v) Inner and outer products are always performed before geometric products. This enables us to remove unnecessary brackets. For example, the expression $a \cdot b\, c$ is to be read as $(a \cdot b)c$.

(vi) Angled brackets $\langle M \rangle_p$ are used to denote the result of projecting onto the terms in M of grade p. The subscript zero is dropped for the projection onto the scalar part.

(vii) The reverse of the multivector M is denoted either with a dagger, $M^\dagger$, or with a tilde, $\tilde{M}$. The latter is employed for applications in spacetime.

(viii) Linear functions are written in an upright font as $\mathsf{F}(a)$ or $\mathsf{h}(a)$. This helps to distinguish linear functions from multivectors. Some exceptions are encountered in chapters 13 and 14, where caligraphic symbols are used for certain tensors in gravitation. The adjoint of a linear function is denoted with a bar, $\bar{\mathsf{h}}(a)$.

(ix) Lie groups are written in capital, Roman font as in $\mathrm{SU}(n)$. The corresponding Lie algebra is written in lower case, $\mathrm{su}(n)$.

Further details concerning the conventions adopted in this book can be found in sections 2.5 and 4.1.