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# Hilbert $C^*$ -Modules

A toolkit for operator algebraists

E.C. Lance  
*University of Leeds*



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## Preface

This book is designed as a “second course in  $C^*$ -algebras”. It presupposes a familiarity with the elementary theory of  $C^*$ -algebras (the GNS construction, the functional calculus) such as may be found in any of the several excellent texts now available—[Dix 2], [KadRin], [Mur], [Ped], [Tak], for example. The aim, as indicated by the subtitle, is to provide the student with a collection of techniques that have shown themselves to be useful in a variety of contexts in modern  $C^*$ -algebra theory.

These techniques centre round the quite elementary and natural concept of a Hilbert  $C^*$ -module. As explained in Chapter 1, this is an object like a Hilbert space except that the inner product is not scalar-valued, but takes its values in a  $C^*$ -algebra. The first three chapters present the elementary theory of Hilbert  $C^*$ -modules and their bounded adjointable operators. From Chapter 4 onwards, tensor products figure prominently, and some knowledge of tensor products of  $C^*$ -algebras (summarised at the beginning of Chapter 4) is needed.

Hilbert  $C^*$ -modules have had three main areas of applications:

- the work of Rieffel and others on induced representations and Morita equivalence ([BroGreRie], [Rie 1], [Rie 2]);
- the work of Kasparov and others on KK-theory ([BaaJul], [Kas 1], [Kas 2]);
- the work of Woronowicz and others on  $C^*$ -algebraic quantum group theory ([BaaSka], [Wor 5]).

There is not very much information on any of these topics in this book, since the aim is to develop a toolkit rather than to demonstrate the tools in use. However, the choice of topics has been made very much with applications in mind, and I have deliberately omitted some aspects of the theory of Hilbert  $C^*$ -modules that seem to me to be internally rather than externally motivated. Broadly speaking, Chapters 4 to 6 are oriented towards KK-

theory, and Chapters 8 onwards towards quantum groups, with Chapter 7 providing a short interlude on the topic of Morita equivalence. (There is, however, considerable overlap between the applications. For example, the theory of unbounded operators on Hilbert C\*-modules was developed in [BaaJul] for its applications to KK-theory, and then rediscovered in [Wor 5] in connection with non-compact quantum groups.)

Readers wishing to see how Hilbert C\*-modules are used in KK-theory are now well served by textbooks. As well as the overview given in [Bla] and the more technical account in [JenTho], there is a very accessible and reader-friendly account of K-theory in [Weg]. Wegge-Olsen's book, which appeared after the first draft of this book had been written, includes a useful chapter on Hilbert C\*-modules. Indeed, there is some overlap (though not too much) between the earlier chapters of this book and Chapter 15 of [Weg]. Wegge-Olsen also devotes a chapter to the beautiful theorem of Cuntz, Higson and Mingo ([CunHig], [Min]) that if  $A$  is a  $\sigma$ -unital C\*-algebra then the unitary group of  $\mathcal{L}(H_A)$  is contractible. I was dissuaded from including a proof of this result only by the fact that it requires more K-theoretic preliminaries than I was prepared to deal with.

The primary goal of these notes, however, is to orient the reader towards the literature on C\*-algebraic quantum groups. To that end, there is a fairly careful account in Chapter 8 of the bialgebra structure of  $C_r^*(G)$ , the reduced C\*-algebra of a locally compact group  $G$ . Quantum groups as such do not figure in this book at all, but there is a short concluding chapter which tries to point the student towards some of the exciting current developments in this area, many of which require some knowledge of Hilbert C\*-modules.

There are no exercises at the end of the chapters, but there are many places throughout the book where details (and sometimes more substantial proofs) are omitted, with or without an invitation to the reader to complete the argument. In a more general sense, the entire book (like any research-level text) should be seen as an exercise for the reader, who should approach it in a critical and participatory frame of mind, and not just as a passive recipient of knowledge.

The list of References has been kept as short as possible, maybe too short. I have cited at the end of each chapter my main (conscious) primary sources, but I may have omitted to acknowledge the work of others that I

have unconsciously absorbed. To them I apologise. The restricted list of References is in line with my belief that the theory of Hilbert  $C^*$ -modules should be treated as a toolkit for use in applications, rather than as an end in itself. I have, however, benefited greatly from a comprehensive unpublished bibliography on Hilbert  $C^*$ -modules produced by M. Frank in 1991 (preprint, University of Leipzig). When giving references for standard results in  $C^*$ -algebra theory, I have tried when possible to cite at least two of the texts mentioned in the opening paragraph above, in the hope that students will learn to consult a variety of sources and not to rely on any one book, however inspired, as their bible.

These notes are a slightly expanded form of a set of ten lectures given at a Summer School in Trondheim, August 1993. I am grateful to NorFA, the Nordic Academy for Advanced Study, for financial support, and to T. Digernes, M.B. Landstad, C. Skau and the graduate student participants in the Summer School for a most enjoyable meeting. The first draft of the book was mostly written during a sabbatical visit to the University of Pennsylvania, for which I am grateful to R.V. Kadison. I am indebted to several people (notably S. Eschmeier, M. Frank and N. Wegge-Olsen) for their helpful comments and corrections. Finally, I should like to thank Mrs A. Landford for her rapid and efficient typing of the  $\text{\TeX}$ script.

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