Chapter 1

Modules and mappings

Hilbert C*-modules first appeared in the work of Kaplansky [Kap], who used them to prove that derivations of type I AW*-algebras are inner. His idea, as explained in the introduction to [Kap], was to generalise Hilbert space by allowing the inner product to take values in a (commutative, unital) C*-algebra rather than in the field of complex numbers. Before making the formal definition, let us briefly consider why this idea might be useful.

Let A be a commutative, unital C*-algebra. By the commutative Gelfand–Naimark theorem we can identify A with $C(X)$, the algebra of continuous complex-valued functions on a compact Hausdorff space X. If X were a euclidean manifold then one would analyse it by geometric techniques, among the most important of which is the study of vector bundles over X. A vector bundle E can be described as follows. Take a fixed euclidean space H, and for each t in X let H_t be a subspace of H. Let E be the space of all continuous functions ξ from X to H such that, for all t in X, $\xi(t) \in H_t$. Then E is naturally endowed with a $C(X)$-valued inner product. Namely, if $\xi, \eta \in E$ then we define $\langle \xi, \eta \rangle$ to be the function

$$t \mapsto \langle \xi(t), \eta(t) \rangle_H.$$

Also, E has the structure of a $C(X)$-module: given ξ in E and f in $C(X)$, we define ξf to be the pointwise product $t \mapsto \xi(t)f(t)$, which is an element of E.

Since vector bundles are so effective in the study of manifolds, it is natural to want to extend the above construction to a general compact

Hausdorff space X. This time, we take H to be a Hilbert space, and for each t in X we ask that H_t should be a closed subspace of H. The construction works exactly as before, and gives rise to a $C(X)$-module E equipped with a $C(X)$-valued inner product. This is the prototypical example of a Hilbert $C(X)$-module.

Explaining his decision to work only with modules over commutative unital C*-algebras, Kaplansky wrote in a footnote to [Kap]: "The assumption of a unit element is not vital, but it seems pointless to omit it, since A will shortly be an AW*-algebra. On the other hand, extension of the theory to modules over non-commutative C*-algebras presents many difficulties." Perhaps because of this discouraging observation, there was essentially no more work on Hilbert C*-modules for a twenty-year period until the PhD thesis of Paschke [Pas]. Paschke showed that, contrary to Kaplansky's misgivings, most of the basic properties of Hilbert C*-modules are valid for modules over an arbitrary C*-algebra. At about the same time, Rieffel [Rie 1] independently developed much of the same theory, and used Hilbert C*-modules as the technical basis for his theory of induced representations of C*-algebras.

Since then the subject has grown and spread rapidly. Many of the most incisive developments were made by Kasparov, who used Hilbert C*-modules as the framework for his bivariant K-theory. Much of this book will consist of an exposition of Kasparov's work. More recently, Hilbert C*-modules have formed the technical underpinning for the C*-algebraic approach to quantum groups. The later chapters of this book are designed to orient the reader towards the literature on this subject.

We now make the formal definition of the objects that we shall be studying. Let A be a C*-algebra (not necessarily unital or commutative). An *inner-product A-module* is a linear space E which is a right A-module (with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in E$, $a \in A$, $\lambda \in \mathbb{C}$), together with a map $(x, y) \mapsto \langle x, y\rangle : E \times E \to A$ such that

$$\left.\begin{array}{lll}
\text{(i)} & \langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle & (x, y, z \in E,\ \alpha, \beta \in \mathbb{C}),\\
\text{(ii)} & \langle x, ya\rangle = \langle x, y\rangle a & (x, y \in E,\ a \in A),\\
\text{(iii)} & \langle y, x\rangle = \langle x, y\rangle^* & (x, y \in E),\\
\text{(iv)} & \langle x, x\rangle \geqslant 0; \quad \text{if } \langle x, x\rangle = 0 \text{ then } x = 0. &
\end{array}\right\} \qquad (1.1)$$

Note that condition (i) requires the inner product to be linear in its *second* variable. It follows from (iii) that the inner product is conjugate-linear in its first variable. We adopt the same convention for ordinary inner-product spaces and Hilbert spaces (so that an inner-product space is the same thing as an inner-product $\mathbb{C}$-module). This convention is in line with much of the recent research literature; but the reader should be aware that many authors use inner products that are linear in the first variable (and conjugate-linear in the second variable).

If E satisfies all the conditions for an inner-product A-module except for the second part of (iv) then we call E a *semi-inner-product A-module*. For such modules there is a useful version of the Cauchy–Schwarz inequality:

PROPOSITION 1.1. *If E is a semi-inner-product A-module and $x, y \in E$ then*

$$\langle y, x\rangle\langle x, y\rangle \leqslant \|\langle x, x\rangle\| \, \langle y, y\rangle. \tag{1.2}$$

Proof. Suppose, as we may without loss of generality, that $\|\langle x, x\rangle\| = 1$. For $a \in A$, we have

$$\begin{aligned} 0 &\leqslant \langle xa - y, xa - y\rangle \\ &= a^*\langle x, x\rangle a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle \\ &\leqslant a^*a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle. \end{aligned}$$

(The last line comes from the fact (1.6.8 in [Dix 2], or 1.3.5 in [Ped]) that if c is a positive element of A then $a^*ca \leqslant \|c\| a^*a$.) Now put $a = \langle x, y\rangle$ to get $a^*a \leqslant \langle y, y\rangle$, as required.

For x in E we write $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. It follows from Proposition 1.1 that $\|\langle x, y\rangle\| \leqslant \|x\| \, \|y\|$ and it is easy to deduce from this that if E is an inner-product A-module then $\|\cdot\|$ is a norm on E. If E is just a semi-inner-product A-module then $\|\cdot\|$ is a seminorm on E and, exactly as for ordinary inner-product spaces, we can construct a quotient of E that is an inner-product A-module, using Proposition 1.1. In fact, let

$$N = \{x \in E : \langle x, x\rangle = 0\}.$$

Then N is a sub-A-module of E. (It is closed under addition by Proposition 1.1.) There is a well-defined A-valued inner product on the quotient

A-module E/N given by

$$\langle x+N, y+N\rangle = \langle x,y\rangle \qquad (x,y\in E)$$

and this makes E/N into an inner-product A-module.

As well as its scalar-valued norm $\|\cdot\|$, an inner-product A-module E has an A-valued "norm" $|\cdot|$, given by $|x| = \langle x,x\rangle^{\frac{1}{2}}$. Since taking square roots of positive elements is an order-preserving operation in a C*-algebra (see 2.2.6 in [Mur] or 1.3.8 in [Ped]), it follows from Proposition 1.1 that

$$|\langle x,y\rangle| \leqslant \|x\|\,|y| \qquad (x,y\in E) \tag{1.3}$$

(where, for $a\in A$, $|a| = (a^*a)^{\frac{1}{2}}$). Notice that the norm on E makes E into a normed A-module. That is to say,

$$\|xa\| \leqslant \|x\|\,\|a\| \qquad (x\in E,\ a\in A). \tag{1.4}$$

Reason: $\langle xa,xa\rangle = a^*\langle x,x\rangle a \leqslant \|x\|^2 a^*a$. Taking square roots, we have $|xa| \leqslant \|x\|\,|a|$, from which the result follows.

The A-valued norm on an inner-product A-module is a useful device, but it needs to be handled with care. For example, it need not be the case that $|x+y| \leqslant |x| + |y|$.

An inner-product A-module which is complete with respect to its norm is called a *Hilbert A-module*, or a *Hilbert C*-module over the C*-algebra A*. Given an (incomplete) inner-product A-module E_0, one can form its completion E just as in the case of an ordinary inner-product space, and thus obtain a Hilbert A-module. This construction makes use of the completeness of the C*-algebra A: for given sequences (x_n), (y_n) in E_0 with limits x, y in E, we want to define $\langle x,y\rangle$ to be $\lim_{n\to\infty}\langle x_n,y_n\rangle$, and the completeness of A is needed to ensure that this limit exists.

Let A_0 be a pre-C*-algebra. That is to say, A_0 satisfies all the conditions to be a C*-algebra except that it need not be complete. Then A_0 has a completion A which is a C*-algebra. We can define an inner-product A_0-module in exactly the same way as an inner-product A-module, and everything that we have done so far still works, except for the completion process in the previous paragraph. In particular, if E is a Hilbert A_0-module then the inequality (1.4) enables us to extend the module action of A_0 on E by continuity to a module action of A on E and thus to make E into a Hilbert A-module.

We now want to combine the two kinds of completion that we have been considering. Suppose that A_0 is a pre-C*-algebra and that E_0 is an inner-product A_0-module. Let A, E be the completions of A_0, E_0. Then E is a module over A_0 and it can be equipped with an inner product taking values in A. We can, however, extend the module action of A_0 on E to an action of A on E so that, finally, E is a Hilbert A-module.

Let E be a Hilbert A-module and let (e_i) be an approximate unit for A. For x in E,

$$\langle x - xe_i, x - xe_i\rangle = \langle x,x\rangle - e_i\langle x,x\rangle - \langle x,x\rangle e_i + e_i\langle x,x\rangle e_i \xrightarrow{i} 0. \qquad (1.5)$$

This shows that EA is dense in E. It also shows that $x1 = x$ if A has an identity 1. If A is not unital and A^+ denotes the C*-algebra obtained by adjoining an identity 1 to A then E becomes a Hilbert A^+-module if we define $x1 = x \quad (x \in X)$.

Define $\langle E,E\rangle$ to be the linear span of the set $\{\langle x,y\rangle : x,y \in E\}$. Then the closure of $\langle E,E\rangle$ is a two-sided ideal in A, call it B. If (u_i) is an approximate unit for B then the calculation in the previous paragraph shows that $xu_i \xrightarrow{i} x$ for all x in E. It follows that $E\langle E,E\rangle$ is dense in E, a fact that will have very useful consequences later. Note that B need not be the whole of A. For example, take $A = C(X)$ and let E be the Hilbert A-module described at the beginning of the chapter. If Y is a nonempty closed subset of X and $H_t = \{0\}$ whenever $t \in Y$ then it is easily seen that $B \subseteq \{f \in A : f(Y) = \{0\}\}$, which is a proper ideal of A.

It is time to have some more examples of Hilbert C*-modules. If A is a C*-algebra, then A itself is a Hilbert A-module if we define

$$\langle a,b\rangle = a^*b \qquad (a,b \in A).$$

If J is a closed right ideal in A then J is a sub-A-module of A and is therefore a Hilbert A-module.

If $\{E_i\}$ is a finite set of Hilbert A-modules then we can form the direct sum $\bigoplus E_i$. This is an A-module in the obvious way, and it becomes a Hilbert A-module if we define $\langle x,y\rangle = \sum_i \langle x_i,y_i\rangle$, where $x = (x_i)$ and $y = (y_i)$. We write E^n for the direct sum of n copies of a Hilbert A-module E.

Now let $\{E_i\}_{i\in I}$ be an infinite set of Hilbert A-modules. The construction of the direct sum in this case is more subtle, and needs to be taken carefully. We define $\bigoplus E_i$ to be the set of all sequences $x = (x_i)$, with x_i in E_i, such that $\sum_i \langle x_i, x_i\rangle$ converges in A. Note that this is a weaker condition than requiring that the series of norms $\sum_i \|\langle x_i, x_i\rangle\|$ should converge (because a series $\sum_n a_n$ of positive elements of a C*-algebra can converge without $\sum_n \|a_n\|$ converging). For $x = (x_i)$ and $y = (y_i)$ in $\bigoplus E_i$, we define $\langle x, y\rangle = \sum_i \langle x_i, y_i\rangle$. To see that this sum converges in A, note that if J is any finite subset of I then Proposition 1.1 applied to the finite direct sum $\bigoplus_J E_i$ shows that

$$\Big\|\sum_{i\in J}\langle x_i, y_i\rangle\Big\|^2 \leqslant \Big\|\sum_{i\in J}\langle x_i, x_i\rangle\Big\| \, \Big\|\sum_{i\in J}\langle y_i, y_i\rangle\Big\|.$$

Since x and y are in $\bigoplus E_i$, the right-hand side of this inequality can be made small whenever J is disjoint from some finite subset of I. This is what is needed to show that the inner product on $\bigoplus E_i$ is well-defined; and this inner product evidently makes $\bigoplus E_i$ into an inner-product A-module. In fact, it is complete, and is therefore a Hilbert A-module. (Exercise: prove this.)

If H is a Hilbert space then the algebraic (vector space) tensor product $H \otimes_{\text{alg}} A$ (which is a right A-module) has an A-valued inner product given on simple tensors by

$$\langle \xi\otimes a, \eta\otimes b\rangle = \langle \xi, \eta\rangle a^* b \qquad (\xi, \eta \in H,\ a, b \in A).$$

(To see that the inner product given in this way is positive definite, write $t = \big\langle \sum \xi_i\otimes a_i, \sum \xi_i\otimes a_i\big\rangle$, and let $\{\varepsilon_k\}$ be an orthonormal basis for the finite-dimensional subspace of H spanned by the ξ_i. If $\xi_i = \sum_k \lambda_{ik}\varepsilon_k$ then $t = \sum_k \big(\sum_i \lambda_{ik} a_i\big)^* \big(\sum_i \lambda_{ik} a_i\big)$. So $t \geqslant 0$; and if $t = 0$ then $\sum_i \lambda_{ik} a_i = 0$ for all k, from which $\sum_i \xi_i\otimes a_i = 0$.) Thus $H \otimes_{\text{alg}} A$ is an inner-product A-module, and we denote its completion by $H \otimes A$. If $\{\varepsilon_i\}$ is an orthonormal basis for H then $H \otimes A$ can be naturally identified with the Hilbert A-module $\bigoplus A_i$ defined in the previous paragraph, where each A_i is a copy of A. In the case where H is a separable, infinite-dimensional Hilbert space, the Hilbert A-module $H \otimes A$ is often denoted by H_A. It plays an important special role in the theory, as we shall see in subsequent chapters.

For a final example, suppose that A is a unital C*-algebra, B is a C*-subalgebra of A containing the identity, and $\psi: A \to B$ is a linear norm-reducing idempotent. Such a map is called a conditional expectation from A to B. A conditional expectation is always a positive map and satisfies

$$\psi(bac) = b\psi(a)c \qquad (a \in A,\ b, c \in B)$$

(see [Tak]). It is said to be faithful if

$$a \in A,\ a \geqslant 0,\ \psi(a) = 0 \implies a = 0.$$

Suppose that E is a Hilbert A-module, with inner product $\langle \cdot, \cdot \rangle_A$. Then E is a semi-inner-product B-module under the inner product given by

$$\langle x, y \rangle_B = \psi(\langle x, y \rangle_A).$$

If ψ is faithful then E is an inner-product B-module.

The above construction is called *localisation*. It will be studied further in Chapter 5 (where we shall eliminate the condition that A and B should be unital).

One would like to think that Hilbert C*-modules behave like Hilbert spaces, and in some ways they do. For example, if E is a Hilbert A-module and $x \in E$ then it is easy to check that

$$\|x\| = \sup\{\|\langle x, y \rangle\| : y \in E,\ \|y\| \leqslant 1\}$$

(a fact that we shall frequently use). But there is one fundamental way in which Hilbert C*-modules differ from Hilbert spaces. Given a closed submodule F of a Hilbert A-module E, define

$$F^\perp = \{y \in E: \langle x, y \rangle = 0 \ \ (x \in F)\}.$$

Then $F^\perp$ is also a closed submodule of E. But E is not (usually) equal to $F \oplus F^\perp$ (and $F^{\perp\perp}$ is usually larger than F). For example, take $A = C(X)$ and let Y be a nonempty closed subset of X whose complement is dense in X. Let $E = A$ and let $F = \{f \in A: f(Y) = \{0\}\}$. In this case, $F^\perp = \{0\}$.

Since the whole theory of Hilbert spaces and their operators is based on the use of orthogonal complements, it is clear that there will be obstacles to developing an analogous theory for Hilbert C*-modules. Nevertheless, it is useful to use Hilbert space ideas as a guide, adding extra conditions when necessary to obtain a theory which works for Hilbert C*-modules. With this in mind, we now introduce some important classes of operators on Hilbert C*-modules.

Suppose that E, F are Hilbert A-modules. We define $\mathcal{L}(E,F)$ to be the set of all maps $t: E \to F$ for which there is a map $t^*: F \to E$ such that

$$\langle tx, y\rangle = \langle x, t^*y\rangle \qquad (x \in E,\ y \in F).$$

It is easy to see that t must be A-linear (that is, t is linear and $t(xa) = t(x)a$ for all $x \in E$, $a \in A$). For each x in the unit ball E_1 of E, define $f_x: F \to A$ by

$$f_x(y) = \langle tx, y\rangle \qquad (y \in F).$$

Then $\|f_x(y)\| \leqslant \|t^*y\|$ for all x in E_1. It follows from the Banach–Steinhaus theorem that the set $\{\|f_x\|: x \in E_1\}$ is bounded, and this shows that the mapping t is bounded. We call $\mathcal{L}(E,F)$ the set of *adjointable* maps from E to F.

Thus every element of $\mathcal{L}(E,F)$ is a bounded A-linear map. It is important to realise that the converse is false: a bounded A-linear map need not be adjointable. For example, let X be a compact Hausdorff space and let Y be a closed nonempty subset of X with dense complement. Let $F = A = C(X)$, let $E = \{f \in A: f(Y) = \{0\}\}$ and let $i: E \to F$ be the inclusion mapping. A simple calculation shows that if i were adjointable and if 1 denotes the identity element of A then $i^*(1)$ would have to be equal to 1. But $1 \notin E$, and so i cannot be adjointable.

It is clear that if $t \in \mathcal{L}(E,F)$ then $t^* \in \mathcal{L}(F,E)$. If also G is a Hilbert A-module and $s \in \mathcal{L}(F,G)$ then $st \in \mathcal{L}(E,G)$. In particular, $\mathcal{L}(E,E)$, which we abbreviate to $\mathcal{L}(E)$, is a $*$-algebra. In fact, it is a C*-algebra. For it is a closed subset of the algebra of all bounded operators on E, and therefore a Banach algebra; and the calculation

$$\begin{aligned}\|t^*t\| &\geqslant \sup\{\|\langle t^*tx, x\rangle\| : x \in E_1\}\\ &= \sup\{\|\langle tx, tx\rangle\| : x \in E_1\} = \|t\|^2\end{aligned}$$

shows that the operator norm satisfies the C*-condition.

It will occasionally be convenient to use the notations $\mathcal{L}_A(E)$, $\mathcal{L}_A(E,F)$ in place of $\mathcal{L}(E)$, $\mathcal{L}(E,F)$, to make explicit the underlying C*-algebra. This will be especially important when, as sometimes happens, we are dealing with a space E that can be a Hilbert C*-module over more than one C*-algebra.

Proposition 1.2. *If $t \in \mathcal{L}(E,F)$ and $x \in E$ then $|tx|^2 \leqslant \|t\|^2\,|x|^2$ and $|tx| \leqslant \|t\|\,|x|$.*

Proof. Let ρ be a state of A. By repeated application of the Cauchy–Schwarz inequality to the semi-inner product $\rho(\langle\cdot\,,\,\cdot\rangle)$ on E, we obtain

$$\begin{aligned}\rho(\langle t^*tx, x\rangle) &\leqslant \rho(\langle t^*tx, t^*tx\rangle)^{\frac12}\rho(\langle x,x\rangle)^{\frac12}\\ &= \rho(\langle (t^*t)^2x, x\rangle)^{\frac12}\rho(|x|^2)^{\frac12}\\ &\leqslant \rho(\langle (t^*t)^2x, (t^*t)^2x\rangle)^{\frac14}\rho(|x|^2)^{\frac12+\frac14}\\ &\;\vdots\\ &\leqslant \rho(\langle (t^*t)^{2^n}x, x\rangle)^{2^{-n}}\rho(|x|^2)^{\frac12+\frac14+\cdots+2^{-n}}\\ &\leqslant (\|x\|^2)^{2^{-n}}\|t^*t\|\,\rho(|x|^2)^{1-2^{-n}}.\end{aligned}$$

As $n \to \infty$ we obtain $\rho(\langle tx, tx\rangle) \leqslant \|t\|^2\rho(|x|^2)$. Since this holds for all states ρ of A, we have $|tx|^2 \leqslant \|t\|^2\,|x|^2$. The second inequality follows on taking square roots.

We now introduce a class of operators analogous to the finite-rank operators on a Hilbert space. Let E, F be Hilbert A-modules. For x in E and y in F, define $\theta_{x,y}\colon F \to E$ by

$$\theta_{x,y}(z) = x\langle y, z\rangle \qquad (z \in F).$$

It is easy to check that $\theta_{x,y} \in \mathcal{L}(F,E)$, with $(\theta_{x,y})^* = \theta_{y,x}$, and also that the following relations hold (where G is a Hilbert A-module):

$$\left.\begin{aligned}\theta_{x,y}\theta_{u,v} = \theta_{x\langle y,u\rangle,v} &= \theta_{x,v\langle u,y\rangle} && (u \in F,\ v \in G),\\ t\theta_{x,y} &= \theta_{tx,y} && (t \in \mathcal{L}(E,G)),\\ \theta_{x,y}s &= \theta_{x,s^*y} && (s \in \mathcal{L}(G,F)).\end{aligned}\right\}\qquad(1.6)$$

We denote by $\mathcal{K}(F,E)$ the closed linear subspace of $\mathcal{L}(F,E)$ spanned by $\{\theta_{x,y}: x \in E, y \in F\}$, and we write $\mathcal{K}(E)$ for $\mathcal{K}(E,E)$. It follows from the above relations that $\mathcal{K}(E)$ is an ideal (by which we always mean a closed, two-sided ideal unless otherwise indicated) in $\mathcal{L}(E)$. Elements of $\mathcal{K}(F,E)$ are often referred to as "compact" operators. But considered as operators between the Banach spaces F and E they need not be compact, so we shall avoid this terminology. (Example: Let A be a unital C*-algebra. Then $\theta_{1,1}$ is the identity operator on the Hilbert A-module A. So the identity operator is in $\mathcal{K}(A)$. But it is not a compact operator on the Banach space A, unless A is finite-dimensional.)

In particular examples, $\mathcal{K}(E)$ is sometimes quite easy to describe. For example, in the case when $E = A$ we have $\mathcal{K}(E) \cong A$, the isomorphism being given by identifying $\theta_{a,b}$ with the operation of left multiplication by ab^*. We shall frequently make use of this isomorphism, and the reader is urged to establish its existence carefully. The proof will make use of the fact that products are dense in any C*-algebra (this follows from the existence of an approximate unit). If A is unital then $\mathcal{K}(A) = \mathcal{L}(A)$, since it is easily verified that any t in $\mathcal{L}(A)$ consists of left multiplication by $t(1)$. When A is not unital, $\mathcal{L}(A)$ is in general much bigger than $\mathcal{K}(A)$, as we shall see in the next chapter.

If H is a Hilbert space and $\xi, \eta \in H$ then we denote by $\xi \cdot \eta$ the rank-one operator

$$\zeta \mapsto \xi\langle \eta, \zeta \rangle \qquad (\zeta \in H).$$

For the Hilbert A-module $H \otimes A$, we have $\mathcal{K}(H \otimes A) \cong \mathcal{K}(H) \otimes A$, where $\mathcal{K}(H)$ is the C*-algebra of compact operators on H and $\mathcal{K}(H) \otimes A$ denotes the C*-algebraic tensor product of $\mathcal{K}(H)$ and A. The algebraic component of this assertion is easily verified: the correspondence is given by identifying $\theta_{\xi \otimes a, \eta \otimes b}$ with $(\xi \cdot \eta) \otimes ab^*$ where, as before, ab^* stands for left multiplication by ab^*. (Writing c for ab^*, we note for future use that $\mathcal{K}(H \otimes A)$ is generated by elements of the form $(\xi \cdot \eta) \otimes c$.) The analytic component of the proof, in which the norm on $\mathcal{K}(H \otimes A)$ is identified with the C*-tensor product norm on $\mathcal{K}(H) \otimes A$, is deferred until Chapter 4, where we shall study tensor products systematically.

If E, F are Hilbert A-modules then $\mathcal{K}(E^m, F^n)$ can be identified with the set of $m \times n$ matrices over $\mathcal{K}(E, F)$. If