# 0

# Introduction

## 0.1 General Introduction

The theory of multilinear maps on a von Neumann algebra is developed in these notes and applied to the continuous Hochschild cohomology of von Neumann algebras. The methods used are those of von Neumann algebras and complete boundedness rather than of homological algebra, and only elementary cohomlogical techniques are employed in the proofs. We have chosen to base our presentation on the problem of whether the continuous cohomology groups $H^n(\mathcal{M},\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ over itself are zero for all $n$. This, and closely related questions, has stimulated much of the recent development of the theory of completely bounded maps, and so we have adopted an approach which has wider applications beyond cohomology theory. The results in these notes have been proved in full generality, provided that they do not stray too far from the central topic of dual normal modules over von Neumann algebras.

There are two main reasons for investigating the Hochschild cohomology groups of operator algebras. When they are non-zero they provide invariants which can distinguish classes of algebras; when they are zero they lead to positive results on the stability of algebraic structures and on the space of bounded derivations on an operator algebra. Elliott's classification of separable $AF$-$C^*$-algebras by $K$-theory is an example of the use of a homological invariant [El1]. Connes has characterized the injective von Neumann algebras by the vanishing of their cohomology over all dual normal modules [Co1], while a further example is the equivalence of amenability and nuclearity for $C^*$-algebras, established by Haagerup [Ha1]. In each of these latter results cohomological conditions are closely related to metric properties of von Neumann algebras and $C^*$-algebras. This theme will recur subsequently in these notes when geometrical ideas and methods are used to deduce information about cohomology groups.

Before presenting an outline of the contents of these notes, we introduce the Hochschild complex and give a brief history of the subject. The reader will find more detailed historical remarks at the end of each chapter.

## 0.2 The Hochschild Complex

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{X}$ be a Banach $\mathcal{M}$-bimodule. By this we mean that there is a module action of $\mathcal{M}$ on both the left and right of $\mathcal{X}$ satisfying

$$\|mx\| \leq \|m\|\,\|x\| \quad \text{and} \quad \|xm\| \leq \|x\|\,\|m\|$$

for all $m \in \mathcal{M}, x \in \mathcal{X}$. We assume throughout that

$$1x = x1 = x$$

for all $x \in \mathcal{X}$, where 1 is the identity of $\mathcal{M}$. We will be concerned with two main examples. The bimodule may be $\mathcal{M}$ itself, or $B(H)$ if $\mathcal{M}$ is represented on a Hilbert space $H$; in both cases the module action is the standard multiplication of operators. The space of $\mathcal{X}$-valued continuous $n$-linear maps on the $n$-fold Cartesian product $\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}$ is denoted by $\mathcal{L}^n(\mathcal{M}, \mathcal{X})$ for $n \geq 1$, while $\mathcal{L}^0(\mathcal{M}, \mathcal{X})$ is defined to be $\mathcal{X}$.

The coboundary operator $\partial$: $\mathcal{L}^n(\mathcal{M}, \mathcal{X}) \to \mathcal{L}^{n+1}(\mathcal{M}, \mathcal{X})$ is defined as follows. For $n = 0$,

$$(\partial x)(a) = ax - xa \qquad (x \in \mathcal{X}, a \in \mathcal{M}),$$

while for $n \geq 1$,

$$\begin{aligned}(\partial\phi)(a_1, \ldots, a_{n+1}) = {} & a_1\phi(a_2, \ldots, a_{n+1}) \\ & + \sum_{j=1}^{n} (-1)^j \phi(a_1, \ldots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \ldots, a_{n+1}) \\ & + (-1)^{n+1}\phi(a_1, \ldots, a_n)a_{n+1}\end{aligned}$$

where $\phi \in \mathcal{L}^n(\mathcal{M}, \mathcal{X})$ and $a_1, \ldots, a_{n+1} \in \mathcal{M}$. It is easy to check that $\partial^2$: $\mathcal{L}^n(\mathcal{M}, \mathcal{X}) \to \mathcal{L}^{n+2}(\mathcal{M}, \mathcal{X})$ is always zero. For example, if $n = 0$ then

$$\begin{aligned}(\partial^2 x)(a_1, a_2) &= a_1(a_2 x - x a_2) - (a_1 a_2 x - x a_1 a_2) + (a_1 x - x a_1)a_2 \\ &= 0.\end{aligned}$$

For the general case, let $\psi_0, \ldots, \psi_{n+1}$ denote the $(n+1)$-linear maps in the definition of $\partial\phi$, so that

$$\partial\phi = \psi_0 + \sum_{j=1}^{n} \psi_j + \psi_{n+1}.$$

There are six types of terms in the sum for $\partial^2\phi$. We show that they occur in pairs with opposite signs, proving that $\partial^2\phi = 0$:

(i) $a_1 a_2 \phi(a_3, \ldots, a_{n+2})$ occurs twice in $\partial\psi_0$ as

$$a_1\psi_0(a_2, \ldots, a_{n+2}) - \psi_0(a_1 a_2, a_3, \ldots, a_{n+2}),$$

(ii) $a_1\phi(a_2, \ldots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \ldots, a_{n+2})$ occurs with coefficients $(-1)^j$ and $(-1)^{j-1}$ respectively in $\partial\psi_0$ and $\partial\psi_{j-1}$,

(iii) $\phi(a_1,\ldots,a_{j-1},a_ja_{j+1},a_{j+2},\ldots,a_{r-1},a_ra_{r+1},a_{r+2},\ldots,a_{n+2})$ occurs with coefficients $(-1)^{j+r-1}$ and $(-1)^{r+j}$ respectively in $\partial\psi_j$ and $\partial\psi_r$,

(iv) $\phi(a_1,\ldots,a_{j-1},a_ja_{j+1},a_{j+2},a_{j+3},\ldots,a_{n+2})$ occurs with coefficients $(-1)^{2j}$ and $(-1)^{2j+1}$ respectively in $\partial\psi_j$ and $\partial\psi_{j+1}$,

(v) $\phi(a_1,\ldots,a_{n-1},a_na_{n+1})a_{n+2}$ occurs with coefficients $(-1)^{2n+2}$ and $(-1)^{2n+1}$ respectively in $\partial\psi_n$ and $\partial\psi_{n+1}$,

(vi) $\phi(a_1,\ldots,a_n)a_{n+1}a_{n+2}$ occurs twice in $\partial\psi_{n+1}$ as

$$(-1)^{2n+2}\psi_{n+1}(a_1,\ldots,a_n,a_{n+1}a_{n+2})+(-1)^{2n+3}\psi_{n+1}(a_1,a_2,\ldots,a_{n+1})a_{n+2}.$$

The reader may find it helpful to write out the case $n=1$ explicitly. The (continuous) Hochschild complex for $\mathcal{M}$ acting on $\mathcal{X}$ is

$$\mathcal{L}^0(\mathcal{M},\mathcal{X})\xrightarrow{\partial}\mathcal{L}^1(\mathcal{M},\mathcal{X})\xrightarrow{\partial}\mathcal{L}^2(\mathcal{M},\mathcal{X})\xrightarrow{\partial}\cdots$$

where, from above,

$$\operatorname{Im}(\partial\colon\ \mathcal{L}^{n-1}(\mathcal{M},\mathcal{X})\to\mathcal{L}^n(\mathcal{M},\mathcal{X}))\subseteq\operatorname{Ker}(\partial\colon\ \mathcal{L}^n(\mathcal{M},\mathcal{X})\to\mathcal{L}^{n+1}(\mathcal{M},\mathcal{X})).$$

The $n^{\text{th}}$ Hochschild cohomology group $H^n(\mathcal{M},\mathcal{X})$ is then defined to be the quotient vector space (which we regard as an additive abelian group)

$$\frac{\operatorname{Ker}(\partial\colon\ \mathcal{L}^n(\mathcal{M},\mathcal{X})\to\mathcal{L}^{n+1}(\mathcal{M},\mathcal{X}))}{\operatorname{Im}(\partial\colon\ \mathcal{L}^{n-1}(\mathcal{M},\mathcal{X})\to\mathcal{L}^n(\mathcal{M},\mathcal{X}))}$$

for $n\geq 1$. Elements of the kernel are called cocycles and elements of the image are called coboundaries.

These groups contain a considerable amount of information about the von Neumann algebra $\mathcal{M}$. Consider the case $n=1$. Then Ker $\partial$ is the space of maps $\phi\colon\ \mathcal{M}\to\mathcal{X}$ satisfying

$$a_1\phi(a_2)-\phi(a_1a_2)+\phi(a_1)a_2=0\qquad(a_1,a_2\in\mathcal{M})$$

and this is precisely the space of bounded derivations. On the other hand, Im $\partial$ is the space of inner derivations of $\mathcal{M}$ into $\mathcal{X}$, and so $H^1(\mathcal{M},\mathcal{X})$ measures how close derivations are to being inner. The higher-order groups do not have such a familiar interpretation, but nevertheless we will see in Chapter 7 that they play an important role in the structure theory of von Neumann algebras.

In the theory developed here, it is important to pass from the complex $(\mathcal{L}^n(\mathcal{M},\mathcal{X}),\partial)$ to more specialized subcomplexes. A particularly important case is $(\mathcal{L}^n_w(\mathcal{M},\mathcal{X}),\partial)$ where the subscript "$w$" indicates ultraweak to weak* continuity of the $n$-linear maps. This gives rise to normal cohomology groups

$H^n_w(\mathcal{M},\mathcal{X})$, which may be easier to compute. Indeed, a major theme in these notes is to calculate $H^n(\mathcal{M},\mathcal{X})$ indirectly by first proving that

$$H^n(\mathcal{M},\mathcal{X}) \cong H^n_*(\mathcal{M},\mathcal{X})$$

for the cohomology groups of a suitable subcomplex $(\mathcal{L}^n_*(\mathcal{M},\mathcal{X}),\partial)$ and then determining $H^n_*(\mathcal{M},\mathcal{X})$. Another important example is the subcomplex obtained by considering $n$-linear maps which are modular with respect to suitable subalgebras. Such maps arise by averaging over amenable unitary groups of operators, and we will devote considerable time to this crucial topic.

## 0.3 History

In [Kap] Kaplansky asked various questions about the properties of derivations on $C^*$-algebras and von Neumann algebras, which may be interpreted as inquiring if certain cohomology groups are equal, or equal to zero. He then established some results for type $I$ algebras. After preliminary work by Kadison [Ka2], Sakai [S1] showed that all the derivations on a von Neumann algebra $\mathcal{M}$ are inner, which is equivalent to showing that the first continuous cohomology group $H^1(\mathcal{M},\mathcal{M})$ is zero. Of course all of this had been preceded by Hochschild's research on the homology and cohomology of rings and algebras [Ho1, Ho2, Ho3]. Here we limit the discussion to von Neumann algebras and their modules and maps. From 1968 to 1972 Johnson, Kadison and Ringrose [KR2, KR3, JKR, J5] proved a number of technical results from which they deduced the equality of certain cohomology groups associated with different classes of continuous maps from the algebra into suitable Banach modules. In particular they showed that a hyperfinite von Neumann algebra $\mathcal{M}$ is amenable as a von Neumann algebra, in the sense that the first cohomology into all dual normal modules is zero. From this it follows that $H^n(\mathcal{M},\mathcal{M}) = 0$ for all $n \geq 1$ when $\mathcal{M}$ is hyperfinite [JKR]. The last two authors conjectured that these groups are zero for all von Neumann algebras $\mathcal{M}$. This conjecture is still open at the time of writing (1994) and it will be discussed more fully subsequently. Their techniques are still important today, and will be presented in Chapter 3 and Section 5.2.

The current theory has taken shape in the last ten years, but there are several other results prior to the early 1980s which should be mentioned. Connes' characterization of injective von Neumann algebras as those which are amenable [Co1] had already been hinted at in his proof of the equivalence of injectivity and hyperfiniteness [Co2]. There were also some investigations of cohomology with the algebra of compact operators as the module [JP, PopR]. In these notes we will only consider dual normal modules, and we will regard more general modules as being beyond our scope. Developments in the theory of completely bounded maps [ChS1, PS], which were partly motivated by cohomology, led to further progress in calculating the cohomology with the algebra $B(H)$ of bounded operators on a Hilbert space as the module [ChES]. These authors introduced completely bounded cohomology groups, showed that they were zero, and in many cases were able to prove that they were equal to the continuous cohomology groups (see Chapter 6). These results established complete boundedness as fundamental to cohomology theory. It enters the picture when a von Neumann algebra is stable under tensoring with $B(H)$ (the type $I, II_\infty$ and $III$ cases) or with the hyperfinite type $II_1$ factor $\mathcal{R}$ (many examples in the type $II_1$ case). Perhaps the most interesting and least understood situation is that of a type $II_1$ factor $\mathcal{M}$ which is not isomorphic to $\mathcal{M}\overline{\otimes}\mathcal{R}$.

The Hochschild cohomology of Banach algebras developed partly in advance of operator algebra theory and partly behind it. We refer the reader to the memoir by Johnson [J3] and the book by Helemskii [He] for the details. The amenable aspects of von Neumann algebras are discussed in these notes only to the extent that they motivate the calculation of $H^n(\mathcal{M}, \mathcal{V})$ for dual normal modules $\mathcal{V}$ over $\mathcal{M}$. The books of Paterson [P1] and Pier [Pie] or the unpublished notes by Thorpe [Th] contain full discussions of amenability. We have omitted the purely algebraic side of the theory in which no assumptions of continuity are made for the cocycles and coboundaries. For this the reader should consult the work of Wodzicki [Wo].

## 0.4 Averaging

The basic idea behind the calculation of cohomology groups is to take an average of a suitable type. This allows us to reduce a general cocycle to one which has desirable algebraic and continuity properties. For a von Neumann algebra this generally means modularity with respect to some hyperfinite subalgebra and normality with respect to each variable.

To motivate the averaging process, consider a von Neumann algebra $\mathcal{M}$, a subalgebra $\mathcal{A}$ generated by a norm compact unitary subgroup $\mathcal{U}$ with normalized Haar measure $\mu$, and a derivation (1-cocycle) $\delta$ of $\mathcal{M}$ into a Banach module $\mathcal{X}$. If $u, v \in \mathcal{U}$, then

$$\delta(uv) = u\delta(v) + \delta(u)v$$

so

$$\begin{aligned}\delta(v) &= u^*\delta(uv) - u^*\delta(u)v \\ &= v(uv)^*\delta(uv) - u^*\delta(u)v.\end{aligned}$$

Integrating with respect to Haar measure and using invariance leads to

$$\delta(v) = vx_0 - x_0v$$

where $x_0 = \int_{\mathcal{U}} u^*\delta(u)d\mu(u)$. This integral is a norm limit of finite approximating sums so defines an element of $\mathcal{X}$. After subtraction of the inner derivation defined by $x_0$ we obtain a new derivation $\tilde{\delta}$ which vanishes on $\mathcal{A}$. The defining equation for a derivation then gives modularity with respect to $\mathcal{A}$:

$$\tilde{\delta}(am) = a\tilde{\delta}(m) + \tilde{\delta}(a)m = a\tilde{\delta}(m),$$
$$\tilde{\delta}(ma) = m\tilde{\delta}(a) + \tilde{\delta}(m)a = \tilde{\delta}(m)a,$$

for $m \in \mathcal{M}$, $a \in \mathcal{A}$. The classical algebraic case of a finite group is a special case; Haar measure becomes a finite averaging process over the group elements. For general von Neumann algebras we will need to move beyond Haar measure and so $\int_{\mathcal{U}} \dots d\mu(u)$ may denote an average by an invariant mean when $\mathcal{U}$ is an amenable unitary group. In this situation the integral will converge in the weak*-topology, forcing us to require $\mathcal{X}$ to be a dual space. Moreover, to pass from $\mathcal{U}$ to the von Neumann algebra it generates requires ultraweak limits and compatibility between the ultraweak topology of $\mathcal{M}$ and the weak*-topology of the module $\mathcal{X}$. For this reason we restrict attention to dual normal modules: $\mathcal{X}$ is a dual space and the maps

$$m \to mx \quad \text{and} \quad m \to xm$$

are both ultraweak-weak* continuous from $\mathcal{M}$ into $\mathcal{X}$ for each fixed element $x \in \mathcal{X}$.

Averages are usually taken over unitary groups, but we will also discuss averages based on projections in Sections 1.7 and 4.2.

## 0.5 Contents

We give only a brief review of the contents, since each chapter begins with an introduction and ends with notes and remarks.

Chapter 1 is devoted to the theory of completely positive and completely bounded maps, and of the Haagerup tensor product. These are crucial tools from Chapter 4 onwards. The representation theorem for completely bounded multilinear maps is proved in detail using the Haagerup tensor product. We follow the operator space approach of [PS] rather than the original one in [ChS1]. The development of complete boundedness occupies the first six sections, and we conclude the chapter by showing the existence of a useful projection from the space of completely bounded maps into a von Neumann algebra $\mathcal{M}$ onto the space of those maps which are modular with respect to $\mathcal{M}$. We subsequently apply this result to show that derivations on von Neumann algebras are inner (Theorem 2.5.1), and that the completely bounded cohomology of a von Neumann algebra over itself is zero (Theorem 4.3.1).

Chapter 2 gives a short account of the theory of derivations on von Neumann algebras into themselves. This is just intended as an introduction to cohomology since there are good detailed accounts of this elsewhere [Di2, S3]. We restrict ourselves to bounded derivations and go only as far as the Kadison–Sakai theorem [Ka2, S1] that derivations on von Neumann algebras are inner. Many readers will be familiar with this. However, our approach is not the standard one, and the techniques in this chapter will occur subsequently in more complicated form.

The third chapter covers the material developed by Johnson, Kadison and Ringrose in the period 1968–1972. Our account has been strongly influenced by two survey articles by Ringrose [Ri3, Ri6]. The notation is different but the organization is the same. The fundamental idea is to average maps over the amenable unitary group of a suitable $C^*$-subalgebra of the given $C^*$-algebra or von Neumann algebra. The averaged map is then used in elementary, but tricky, calculations to show that cohomology groups associated with module maps over the averaged algebra are equal to the unaveraged groups. Alternatively, we could have chosen the more axiomatic and homological approach due to Craw [Cr1, Cr2]. This works well in the continuous and normal cases, but there are major technical problems in the completely bounded theory. The continuous setting requires the duality between the projective tensor product and the space of continuous bilinear forms. In the completely bounded case, all the Banach spaces must be replaced by operator spaces while retaining the correct dualities and relations. This is now possible, due to the theory of operator spaces and their tensor products as recently developed by Blecher and Paulsen [BP] and Effros and Ruan [ER1–ER5] (but which was not available to Craw). We decided that the axiomatic approach would be less intuitive and would require more operator space theory than we thought appropriate.

In Chapter 4 we show that the completely bounded cohomology of a von Neumann algebra $\mathcal{M}$ over $B(H)$ is always zero. This depends on writing cocycles as products of commutators and using a projection as an average. The corresponding result when $\mathcal{M}$ is the module is deduced from this, using the projection of Section 1.7. The representation of a cocycle as a product of commutators is only available in the completely bounded case. For this reason, all recent progress in the continuous setting depends on reduction to this special situation.

Chapter 5 is concerned with hyperfinite subalgebras and maximal abelian self-adjoint subalgebras (masas) in type $II_1$ von Neumann algebras; this is the technical foundation for the succeeding chapter. Section 5.2 contains the important result that a continuous multilinear map on a von Neumann algebra $\mathcal{M}$ to itself which is modular over the centre can be continuously extended to the $C^*$-algebra generated by $\mathcal{M}$ and a masa in the commutant

$\mathcal{M}'$. The point is that the ultraweak closure of this $C^*$-algebra is a type $I$ von Neumann algebra, which is much easier to handle than $\mathcal{M}$ itself.

The following section is devoted to three results of Popa on subalgebras of type $II_1$ factors. The first two provide a hyperfinite subalgebra $\mathcal{N}$ whose relative commutant $\mathcal{N}' \cap \mathcal{M}$ consists of scalar multiples of the identity. These results provide the subalgebras over which we average in Sections 5.5 and 6.3. The third shows how to obtain a masa in $B(L^2(\mathcal{M}))$ from a Cartan subalgebra in $\mathcal{M}$. In Section 5.4 we introduce the Haagerup–Pisier non-commutative Grothendieck inequality. The normal form of this is used to show that certain bimodule maps lift to the Haagerup tensor product. Combined with results from Section 1.6, this is an important step towards proving that certain cocycles are completely bounded in Sections 6.3 and 6.4.

The main calculations of cohomology groups are presented in Chapter 6. Type $I_\infty, II_\infty$, and $III$ von Neumann algebras are isomorphic to their tensor products with $B(H)$, while many type $II_1$ von Neumann algebras are isomorphic to their tensor product with $\mathcal{R}$. In both cases the extra tensor factor allows us to replace a general cocycle by a completely bounded one, after which the theory of Chapter 4 can be applied to prove that $H^n(\mathcal{M}, \mathcal{M}) = 0$ for these algebras. The type $II_1$ case is still open but we use complete boundedness to determine $H^2(\mathcal{M}, \mathcal{M})$ and $H^3(\mathcal{M}, \mathcal{M})$ when there is a Cartan subalgebra (Section 6.3) and $H^2(\mathcal{M}, \mathcal{M})$ for property $\Gamma$ factors (Sections 6.4). We conclude the chapter with a general result which shows that it is sufficient to consider the conjecture "$H^n(\mathcal{M}, \mathcal{M}) = 0$" only for separably acting von Neumann algebras.

Chapter 7 presents that part of the structure theory of von Neumann algebras which depends on cohomology, and is drawn from [J6, RaT]. The main result is local stability of the product in a von Neumann algebra whenever the second and third cohomology groups vanish. The appendix (Chapter 8) contains a short discussion of bounded group cohomology of a discrete group $G$, and its relationship to the Hochschild cohomology of the Banach algebra $\ell^1(G)$. At this time there is no relation to the cohomology of group $C^*$-algebras and von Neumann algebras, but we have included this chapter in the hope that a connection may become apparent in the future.

## 0.6 Background

We have attempted to make these notes as self-contained as possible, and so we have assumed that the reader has had a first course in operator algebras but little beyond that. Knowledge of the basic theory as set out in the book of Kadison and Ringrose [KR4] (or the standard texts of Dixmier [Di2], Sakai [S2] or Takesaki [T]) would be ideal; however much less is required. The reader who knows the GNS representation, type theory of von Neumann algebras, the double commutant theorem, the Kaplansky density theorem, and the various topologies on a von Neumann algebra is ready to begin. We review briefly a few topics which will recur frequently but would be considered beyond basic knowledge.

A von Neumann algebra $\mathcal{M} \subseteq B(H)$ is said to be ***hyperfinite*** if there is an increasing family of finite dimensional $*$-subalgebras $\mathcal{M}_\lambda$ whose union is ultraweakly dense. If there is a norm one projection of $B(H)$ onto $\mathcal{M}$ then $\mathcal{M}$ is called ***injective***. Connes [Co2] proved that these conditions are equivalent and so we use them interchangeably in the text. We will generally say "hyperfinite" if we want to use the finite dimensional subalgebras, and switch to "injective" when we need the norm one projection.

In Chapter 6 we will require some deeper theory for type $II_1$ factors. These are characterized by the existence of a tracial state $tr$ satisfying

$$tr(xy) = tr(yx)$$

for all elements $x, y$ of the factor $\mathcal{M}$. The GNS construction for $tr$ yields a Hilbert space $L^2(\mathcal{M}, tr)$ (or just $L^2(\mathcal{M})$) on which $\mathcal{M}$ acts by left multiplication. Since

$$tr(xx^*) = tr(x^*x),$$

the map $x \to x^*$ extends to a conjugate linear isometry $J$ on $L^2(\mathcal{M})$ and $J\mathcal{M}J$ is the commutant $\mathcal{M}'$. This representation is the ***standard form*** for $\mathcal{M}$. The Hilbert space norm is denoted $\|\cdot\|_2$ and is defined by

$$\|x\|_2 = (tr(x^*x))^{1/2}.$$

If $\mathcal{N}$ is a subfactor of $\mathcal{M}$ then there always exists a unique trace-preserving conditional expectation $E\colon \mathcal{M} \to \mathcal{N}$, that is, a norm one projection such that

$$tr_{\mathcal{N}}(Ex) = tr_{\mathcal{M}}(x)$$

for all $x \in \mathcal{M}$. These facts about type $II_1$ von Neumann algebras may all be found in [KR4].

Cohomology theory is really the study of multilinear maps of a von Neumann algebra $\mathcal{M}$ into a module $\mathcal{X}$, so we review the classes of maps which will be important. The space of continuous multilinear maps is denoted by

$$\mathcal{L}^n(\mathcal{M}, \mathcal{X}).$$

If we restrict to maps which are modular with respect to a subalgebra $\mathcal{A}$ in the sense that

$$a\phi(x_1, \ldots, x_n) = \phi(ax_1, \ldots, x_n),$$
$$\phi(x_1, \ldots, x_j a, x_{j+1}, \ldots, x_n) = \phi(x_1, \ldots, x_j, ax_{j+1}, \ldots, x_n),$$
$$\phi(x_1, \ldots, x_n a) = \phi(x_1, \ldots, x_n)a$$

for $a \in \mathcal{A}$, $x_j \in \mathcal{M}$, then we adopt the notation

$$\mathcal{L}^n(\mathcal{M}, \mathcal{X}: /\mathcal{A}).$$

We append the subscripts "$w$" and "$cb$" to indicate ultraweak continuity in each variable and complete boundedness respectively. For example, $\mathcal{L}^n_{wcb}(\mathcal{M}, \mathcal{X}: /\mathcal{A})$ denotes the space of ultraweak-weak* continuous (equivalently normal) completely bounded $n$-linear maps of $\mathcal{M}$ into $\mathcal{X}$ which are modular with respect to $\mathcal{A}$. The cohomology groups $H^n_{wcb}(\mathcal{M}, \mathcal{X}: /\mathcal{A})$ are formed from cocycles and coboundaries from $\mathcal{L}^n_{wcb}(\mathcal{M}, \mathcal{X}: /\mathcal{A})$, with obvious similar interpretations of $H^n_{cb}(\mathcal{M}, \mathcal{X})$, $H^n_{w}(\mathcal{M}, \mathcal{X})$ and $H^n_{wcb}(\mathcal{M}, \mathcal{X})$. A general list of notation is included at the end of these notes.