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Edited by Leila Schneps
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# DESSINS D'ENFANT

## (The theory of cellular maps on Riemann surfaces)

A "dessin d'enfant" is a certain simple type of cellular map on a compact orientable topological surface. It can be visualized as a finite set of points on the surface, connected by a finite set of edges, in such a way that the edges and vertices form a connected set (with a certain property detailed below), which cuts the surfaces into open cells. Typical examples are trees on the sphere; the regular solids also form a well-studied category of dessins on the sphere, considered as such as long ago as 1884 in Felix Klein's book *Das Ikosaeder.* Some visual examples are given in Figure 1.

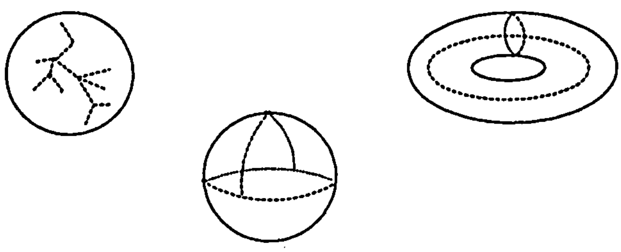

**Figure 1**

Alexander Grothendieck became interested by the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of dessins; this action is highly non-trivial even in the simplest case of genus 0, and depends on a correspondence between dessins and algebraic curves defined over $\overline{\mathbf{Q}}$ which is a direct consequence of Belyi's well-known theorem identifying algebraic curves defined over $\overline{\mathbf{Q}}$ with Riemann surface morphisms $\beta : X \rightarrow \mathbf{P}^1\mathbf{C}$ ramified at most over $\{0, 1, \infty\}$. There are several references on the subject: the necessary basics are contained in the articles by Birch, Schneps, and Bauer-Itzykson in this volume (see their reference lists as well). These facts are at the base of an ambitious program by Grothendieck aiming, among other things, at a complete description of the absolute Galois group. He explains how he was led to this by the problems of research in a provincial university (see the end of the introduction for translations of the French passages):

"Les exigences d'un enseignement universitaire, s'adressant donc à des étudiants (y compris les étudiants dits 'avancés') au bagage mathématique modeste (et souvent moins que modeste), m'ont amené à renouveler de façon draconienne les thèmes de réflexion à proposer à mes élèves, et de fil en aiguille et de plus en plus, à moi-même également. Il m'avait semblé important de partir d'un bagage intuitif commun, indépendant de tout langage technique censé l'exprimer, bien antérieur à tout tel langage – il s'est avéré que l'intuition géométrique et topologique des formes, et plus particulièrement des formes bidimensionnelles, était un tel terrain commun. Il s'agit donc de thèmes qu'on peut grouper sous l'appellation de 'topologie des surfaces' ou 'géométrie des surfaces', étant entendu dans cette dernière appellation que l'accent principal se trouve sur les propriétés topologiques des surfaces, ou sur les aspects combinatoires qui en constituent l'expression technique la plus terre-à-terre, et non sur les aspects différentiels, voire conformes, riemanniens, holomorphes et (de là) l'aspect 'courbes algébriques complexes'. Une fois ce dernier pas franchi cependant, voici soudain la géométrie algébrique (mes anciennes amours!) qui fait irruption à nouveau, et ce par les objets qu'on peut considérer comme les pierres de construction ultimes de toutes les autres variétés algébriques. Alors que dans mes recherches d'avant 1970, mon attention systématiquement était dirigée vers les objets de généralité maximale, afin de dégager un langage d'ensemble adéquat pour le monde de la géométrie algébrique, et que je ne m'attardais sur les courbes algébriques que dans la stricte mesure où cela s'avérait indispensable (notamment en cohomologie étale) pour développer des techniques et énoncés 'passe-partout' valables en toutes dimensions et en tous lieux (j'entends, sur tous schémas de base, voire tous topos annelés de base...), me voici donc ramené, par le truchement d'objets si simples qu'un enfant peut les connaître en jouant, aux débuts et origines de la géométrie algébrique, familiers à Riemann et à ses émules!"[1]

This passage is taken from an unpublished manuscript by Grothendieck called l'*Esquisse d'un Programme*: it dates from 1984 and was written as part of his application for a position in the French Centre National de la Recherche Scientifique for which he applied after many years as a professor at the University of Montpellier, partly out of discouragement with his teaching activities:

"Comme la conjoncture actuelle rend de plus en plus illusoire pour moi les perspectives d'un enseignement de recherche à l'Université, je me suis résolu à demander mon admission au CNRS, pour pouvoir consacrer mon énergie à developper des travaux et perspectives dont il devient clair qu'il ne se trouvera aucun élève (ni même, semble-t-il, aucun congénère mathéticien) pour les développer à ma place."[2]

As indicated by its title, the paper sketches out a vast vision juxtaposing several areas of possible exploration which Grothendieck proposed to study during his tenure in the CNRS, and to write up in an enormous work called *Réflexions Mathématiques*: "il s'agit...de développer des idées et des visions multiples amorcées au cours de ces douze dernières années, en les précisant et les approfondissant, avec tous les rebondissements imprévus qui constamment accompagnent ce genre de travail...et je compte bien laisser apparaître clairement...la démarche de la pensée qui sonde et qui découvre, en tâtonnant dans la pénombre bien souvent, avec des trouées de lumière subite quand quelque tenace image fausse, ou simplement inadéquate, se trouve enfin débusquée et mise à jour, et que les choses qui semblaient de guingois se mettent en place, dans l'harmonie mutuelle qui leur est propre."[3]

One of the most fascinating regions of exploration is contained in sections 2 and 3, where Grothendieck describes an entirely new approach to the problem of describing the structure of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The conference on dessins d'enfant at the CIRM in Luminy, April 19-24, 1993, was organized in an effort to gather together a number of people who were working on or interested in subjects more or less closely related to this part of Grothendieck's *Esquisse d'un Programme*, many of whom were unaware of the fact that others were actively working on the same questions – indeed the *Esquisse d'un Programme* appears to have benefited from a near-universal moratorium among French mathematicians until about a year ago though it was mentioned regularly in articles by Russian and Japanese mathematicians.

The idea of publishing the *Esquisse d'un Programme* in this volume unfortunately had to be abandoned, with regret and reluctance, when it became clear that it was impossible to contact Grothendieck to obtain his written permission to do so. This introduction should serve, in some sense, as a poor substitute, attempting to explain the major underlying ideas of Grothendieck's project to study the absolute Galois group, with a good deal more precision than can be found written in the Esquisse, but certainly much less than could be found in the mind of its author, and with rather more reference to concrete results as presented in various publications including the conference proceedings, but unfortunately a smaller dose of soaring imagination.

* * * * *

In §3 of the Esquisse, entitled *Corps de nombres associé à un dessin d'enfant*, Grothendieck confronts the study of the combinatorical action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the dessins. The main point of departure, as mentioned above, is Belyi's important theorem stating that *every algebraic curve defined over*

*$\overline{\mathbf{Q}}$ can be realized as a covering $\beta : X \to \mathbf{P}^1\mathbf{C}$ where the morphism $\beta$ is a Belyi morphism, namely ramified at most over the points* 0, 1 *and* $\infty$ *of* $\mathbf{P}^1\mathbf{C}$.
This theorem, and in particular the simplicity of its proof, was astounding to Grothendieck who had hardly dared conjecture it: "une telle supposition avait l'air à tel point dingue que j'étais presque gêné de la soumettre aux compétences en la matière. Deligne consulté trouvait la supposition dingue en effet, mais sans avoir un contrexemple dans ses manches. Moins d'un an après, au Congrès International de Helsinki, le mathématicien soviétique Bielyi annonce justement ce résultat, avec une démonstration d'une simplicité déconcertante tenant en deux petites pages d'une lettre de Deligne – jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!"[4]

To understand the question which was answered by Belyi's theorem, we first set out some background. Let the term "dessin d'enfant" (child's drawing), denote a cellular map with a bipartite structure; i.e. a compact topological surface equipped with a finite number of points (vertices) and a finite number of edges connecting them, such that the following properties hold:
i) the set of vertices and edges is connected
ii) this set cuts the surface into open cells
iii) a bipartite structure can be put on the map; it consists of marking the vertices with two different marks, say $\bullet$ and $\star$, in such a way that the direct neighbors of any vertex are all of the opposite mark.

Two special cases of dessins are frequently considered, in the *Esquisse d'un Programme*, in the introductory article [SV], and in the articles in this volume by Schneps and Couveignes-Granboulan for example. The first, which we may call "pre-clean", are those having $\star$ vertices with valencies (i.e. number of edges coming out of them) less than or equal to 2, and the second, known as clean dessins, are those all of whose $\star$ vertices have valency exactly equal to 2. These dessins can be pictured as cellular maps with all vertices labeled $\bullet$ and a $\star$ placed either in the middle or at a tail end of each edge in the pre-clean case, only in the middle in the clean case; they have the advantage of simplicity in that $\bullet$'s correspond to vertices and $\star$'s to edges. Examples of the three types of dessin are given in Figure 2; they are all in genus zero for simplicity, and drawn on the plane, i.e. on the visible part of $\mathbf{P}^1\mathbf{C}$...

To any Belyi function $\beta : X \to \mathbf{P}^1\mathbf{C}$ one can immediately associate a dessin by considering the inverse image $\beta^{-1}[0,1]$ of the real segment $[0,1]$ on $\mathbf{P}^1\mathbf{C}$, drawn on the topological model of the Riemann surface $X$; as a convention we consider $\bullet$'s to represent the pre-images of 0 and $\star$'s of 1 (also, by convention, we mark with a $\circ$ each pre-image of $\infty$, so exactly one

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o falls somewhere into each open cell). Under this association, the pre-clean dessins correspond to Belyi functions whose ramification orders over 1 are all of order less than or equal to 2 and the clean ones to Belyi functions all of whose ramification indices over 1 are exactly equal to 2.

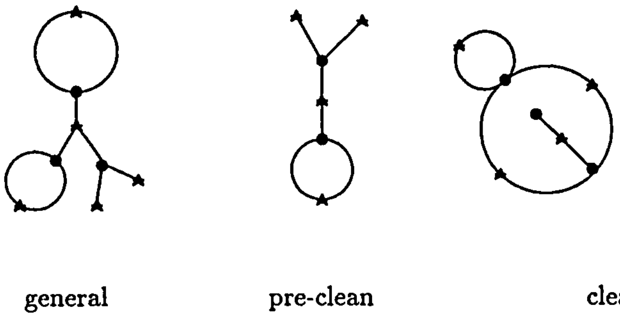


**Figure 2**

Using Belyi's theorem, Grothendieck showed the converse direction, i.e. that to any dessin one can associate a covering $\beta : X \to \mathbf{P}^1\mathbf{C}$ where $\beta$ is a Belyi morphism and $X$ a Riemann surface. A rigorous combinatorical proof of this result is given in a 1975 article by two of Grothendieck's students, J. Malgoire and C. Voisin [MV]; the result was also proved around the same time in an article by D. Singerman and G. Jones ([JS]). The first such calculations were performed by Atkin, who considered dessins as quotients of the Poincaré upper half plane by subgroups of finite index in $PSL_2(\mathbf{Z})$. In their article [SV] in the Grothendieck Festschrift, in the case of clean dessins, Shabat and Voevodsky give a simple topological argument which we describe here as it makes the association intuitively clear. From a dessin $D$, one obtains a canonical triangulation of the surface on which $D$ is drawn as follows. In addition to the $\bullet$'s and the $\star$'s, draw a $\circ$ somewhere inside each open cell, and add edges to the drawing joining every $\circ$ to all the neighboring $\bullet$ and $\star$ points. This paves the surface with lozenges of the type shown in Figure 3 (it forms what is known as the *canonical triangulation* of the surface associated to the dessin, which is discussed in the article by Bauer and Itzykson).

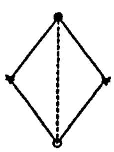

**Figure 3**

Identifying the two sides joining the $\bullet$ to a $\star$, and also the two sides joining the $\circ$ to a $\star$ gives a surface homeomorphic to the sphere. One takes $\beta$ to be the morphism from the topological surface to $\mathbf{P}^1$ sending all these lozenges to $\mathbf{P}^1$ with the $\bullet$ points going to 0, the $\star$'s to 1 and the $\circ$'s to $\infty$. By the Riemann existence theorem, there is a unique complex structure which can be put on the topological surface such that $\beta$ becomes a rational function for it. This gives the covering $\beta : X \to \mathbf{P}^1\mathbf{C}$ associated to the dessin, and the morphism $\beta$ is a Belyi morphism under this construction; its ramification indices over 0, 1 and $\infty$ are given by the number of edges coming out of each $\bullet$, $\star$ and $\circ$ point respectively in the canonical triangulation associated to $D$ (and indeed this triangulation is simply given by $\beta^{-1}(\infty, \infty)$).

Grothendieck had asked the question of whether all algebraic curves defined over $\overline{\mathbf{Q}}$ could be obtained in this way – the affirmative answer is of course a corollary of Belyi's theorem, which gives a natural description – "defined over $\overline{\mathbf{Q}}$" – of the coverings of $\mathbf{P}^1\mathbf{C}$ ramified over just three points. As he describes it: "*il y a une identité profonde entre la combinatoire des cartes finies d'une part, et la géométrie des courbes algébriques définies sur des corps de nombres, de l'autre.* Ce résultat profond, joint à l'interprétation algébrico-géométrique des cartes finies, ouvre la porte sur un monde nouveau, inexploré – et à portée de main de tous, qui passent sans le voir."[5]

With the association of a dessin to any Belyi function described above, we now have a bijection between abstract dessins and isomorphism classes of pairs $(X, \beta)$ where $X$ is a Riemann surface and $\beta$ a Belyi morphism; this then gives rise to a bijection between clean dessins and conjugacy classes of the so-called cartographical group $C_2^+$, which is freely generated by one generator of infinite order and one of order 2. Complete proofs of this, of Belyi's theorem, of the combinatorical proof by Malgoire-Voisin that one can associate a unique isomorphism class of Belyi coverings of $\mathbf{P}^1\mathbf{C}$ to a given dessin and of the topological argument of Shabat and Voevodsky given above; in a word, of the bijection between isomorphism classes of pairs $(X, \beta)$ and abstract dessins, can be found in the first part of the article by Schneps, and also that of Birch in this volume. This bijection gives a natural way to associate a number field $K_D$ to every dessin $D$ by defining it to be the moduli field of the associated algebraic curve $X$ (for a precise discussion of the terms moduli field and field of definition see the articles by Schneps and Couveignes-Granboulan); this field is also called the moduli field of the dessin. The articles of Birch and Malle are concerned with the study of $K_D$; Malle gives tables of information for moduli fields for a number of Belyi coverings of degree $\leq 13$, whereas Birch considers the question of which primes can be ramified there. The latter article also considers the problem, specifically associated to non-congruence subgroups $\Gamma$ of $SL_2(\mathbf{Z})$, of the form of the coefficients of a basis of modular forms for $S^0_{2k}(\Gamma)$, the space of cusp forms of weight $2k$, which was studied among others by Atkin, Swinnerton-Dyer and Scholl.

The difficult aspect of the bijection described above is to render it explicit, in the direction dessins→ coverings (the direction coverings→dessins can be handled via Newton's algorithm for example, cf. part III of Schneps). This can be done, in genus zero on small example, using algebraic methods such as the Gröbner basis algorithem to solve a system of equations: variations of this procedure are described in the articles by Birch, Shabat and Schneps (and [C] for some improvements). In his series of introductory lectures, Oesterlé suggested that the use of the Puiseux series as introduced by Ihara could lead to an explicit analytic description of the coverings, giving a method to calculate the Belyi functions associated to dessins of any genus. Following this suggestion, Couveignes and Granboulan describe what information can be obtained from the Puiseux series method, and how this method can be made more effective in the genus zero case where it can be strengthened by the use of approximations, iteration and intuition. They calculate an "optimal" Belyi function associated to a dessin; namely a function defined over the smallest possible number field, shown to be at worst a quadratic extension of the moduli field of the dessin.

Considering more generally coverings of $\mathbf{P}^1\mathbf{C}$ ramified at $r$ points with $r \geq 3$, the article by Michel Emsalem explains the construction of moduli spaces for these objects, known as Hurwitz spaces: the main theorem states that the Hurwitz spaces are algebraic varieties defined over $\mathbf{Q}$.

Paula Cohen's article is a summary of work by Cohen, Itzykson and Wolfart ([CIW]). To any Belyi morphism $\beta : X \to \mathbf{P}^1\mathbf{C}$, together with the associated triangle group $\Delta(p,q,r)$ where $p,q,r$ are the positive common multiples of the orders of ramification of $\beta$ over 0, 1 and $\infty$, they associate the covering group $H$ which is a subgroup of finite index in $\Delta(p,q,r)$, and which determines an arithmetic group $\Gamma$ which acts on $\mathfrak{H}^t$ for some positive integer $t$ ($\mathfrak{H}$ is the upper half-plane) in such a way that the quotient $V = \Gamma \setminus \mathfrak{H}^t$ is a Shimura variety parametrising isomorphism classes of abelian varieties with certain complex multiplication properties for a subfield $K$ of a cyclotomic field. From the natural injection $H \to \Gamma$ they define an injection $\mathfrak{H} \to \mathfrak{H}^t$ which commutes with the respective group actions; by passing to the quotient, this defines a non-trivial morphism $X \to V$, defined over $\overline{\mathbf{Q}}$, which sends the elements of $\beta^{-1}\{0,1,\infty\}$ onto points of complex multiplication by $K$.

In the article by Jones and Singerman, certain of these triangle groups, such as the cartographical group $C_2^+$ mentioned above, are identified with subgroups of $PGL_2(\mathbb{Z})$: this gives realisations of the associated maps as tessellations of Riemann surfaces or Klein surfaces, and the operations on them of automorphisms of the triangle groups are studied.

Saito's paper gives a real algebraic coordinatization of the Teichmüller space $\mathcal{T}_{g,n}$, the universal covering space of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points, whose fundamental group (as an orbifold) is the mapping class group $M(g,n)$. In his article on horizontal divisors, Ihara studies the geometric ramification properties of Belyi coverings. The article by Bauer and Itzykson contains much of the basic information on dessins d'enfants; this is used to give generating functions in the form of matricial integrals which have applications to intersection problems on the moduli spaces $\mathcal{M}_{g,n}$.

The algorithmic methods described in the articles by Birch, Couveignes-Granboulan, Malle, Schneps and Shabat do allow one to calculate examples of the Galois orbits of a good many genus zero dessins, as a first step towards the exploration of the Galois action. In some sense one should not actually lose information about $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by restricting one's attention to the genus zero dessins, since a result due to H. Lenstra (cf. the article by Schneps, part II) shows that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of genus zero dessins and indeed even on the set of trees which are the simplest example of them. G. Shabat studies the combinatorical properties of trees, and particular the special case of "diameter 4" trees which have several interesting properties, for instance the discriminant of their associated number fields can be studied directly from their combinatorics.

Examples of Galois orbits of genus zero dessins have been much examined with a view to establishing the properties of a given dessin which remain invariant when the Galois group acts on it. Optimistically, one might dream of a "complete" list of such Galois invariants, i.e. a list such that the set of all dessins having the same invariants should constitute exactly one orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Several Galois invariants have been found, such as the valency lists or the Galois group of a dessin. However the list is known not to be complete since two dessins have also been found which do not belong to the same Galois orbit and as yet continue to defy all attempts to separate them via the known Galois invariants. These two dessins, similar diameter 4 trees (see the frontispiece), are given in Figure 4. For the time being, this curious situation would appear to indicate the limits of our understanding...

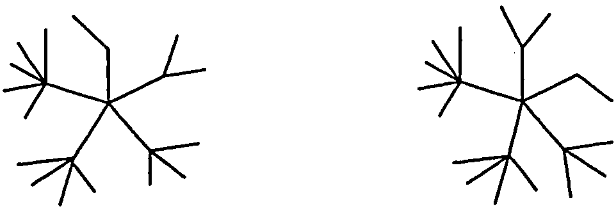

**Figure 4**

Grothendieck was led to the question of the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the dessins via "un effort de compréhension des relations entre groupes de Galois 'arithmétiques' et groupes fondamentaux profinis 'géométriques'. Assez vite il s'oriente vers un travail de formulation calculatoire de l'opération de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ sur $\pi_{0,3}$" [6] (the algebraic fundamental group of $\mathbf{P}^1\mathbf{C} \setminus \{0, 1, \infty\}$, which is just a free profinite group on two generators). Because the cartographical group $C_2^+$ is just a quotient of $\pi_{0,3}$, the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the fundamental group is reflected in its action on the dessins. We mention briefly here that these ideas led him to what he calls the central theme of anabelian algebraic geometry, namely the idea that certain "anabelian" varieties $X$ defined over a $K$ can be entirely reconstituted from knowledge of their "mixed fundamental group", which is the extension of the fundamental group $\pi_1(X_{\overline{K}})$ by $\mathrm{Gal}(\overline{K}/K)$ ($\overline{K}$ denotes the algebraic closure of $K$). Much of this part of Grothendieck's program remains mysterious but some results can be found in work by Bogomolov, Pop, Nakamura, Voevodsky and others (cf. for example [B], [N1], [N2], [NT], [P]).

The space $\mathbf{P}^1\mathbf{C} \setminus \{0, 1, \infty\}$ is the moduli space of Riemann surfaces of genus 0 with 4 marked points. Grothendieck perceived the study of the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $\pi_{0,3}$, the algebraic fundamental group of this moduli space, as a special case of the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on all the fundamental groups of the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points, or indeed, on certain "fundamental groupoids" denoted $\hat{T}_{g,n}$, on a certain finite set of base points. These ideas gave rise to §2 of the Esquisse, entitled *Un "jeu de Légo-Teichmüller", et le groupe de Galois de* $\overline{\mathbf{Q}}$ *sur* $\mathbf{Q}$. Grothendieck considers what he calls the Teichmüller tower of profinite fundamental groupoids $\hat{T}_{g,n}$ of the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ ordered marked points. The groupoids are not explicitly defined in the paper, but many of their properties, including conjectural ones, are enumerated. They should have as "unit group" the usual fundamental groups of these moduli spaces, namely the pure subgroups $K(g, n)$ of the mapping class groups $M(g, n)$, but Grothendieck is adamant on the importance of considering groupoids on several base points:

"...les gens s'obstinent encore, en calculant avec des groupes fondamentaux, à fixer un seul point base, plutôt que d'en choisir astucieusement tout un paquet qui soit invariant par les symétries de la situation, lesquelles sont donc perdues en route."[7]

The set of profinite groupoids $\hat{T}_{g,n}$ should form a *tower* when the different groupoids are linked by certain natural homomorphisms corresponding to degeneration of surfaces. Indeed, surgical operations on the Riemann surfaces which decrease the type $g, n$, such as erasing points, cutting into pieces along geodesics and so on, induce natural morphisms between the fundamental groups or groupoids of the associated moduli spaces (as do the inverse operations of adding marked points and "gluing" pieces together; however these, in an obvious sense, do not merit the term "degeneration"). In his article [D], which although occasionally quite cryptic for the unwary, sheds tremendous illumination on Grothendieck's ideas, Drinfel'd summarizes this idea, writing $(X, x_1, \ldots, x_n)$ for a Riemann surface $X$ with $n$ ordered marked points: "Since degeneration of the set $(X, x_1, \ldots, x_n)$ results in decreasing $g$ and $n$, the groupoids $T_{g,n}$ are connected by certain homomorphisms. The collection of all $T_{g,n}$ and all such homomorphisms is called in [2] the Teichmüller tower."

Needless to say, [2] is the *Esquisse d'un Programme*. Although Drinfel'd's goal in this paragraph is not to study the Teichmüller fundamental groupoids but merely to sketch the relation between the rest of the paper and Grothendieck's ideas, he does make some further brief indications about the choice of base points on the moduli spaces for them. He recommends taking tangential base points "at infinity", i.e. base points in a neighborhood of infinity on the moduli space $M_{g,n}$, corresponding to "maximal degeneration" of Riemann surfaces. These base points generalize to all the moduli spaces the well-known tangential base points defined by Deligne in [De] on $\mathbf{P}^1\mathbf{C} - \{0, 1, \infty\}$, which is the first non-trivial moduli space $M_{0,4}$. This construction of a Teichmüller tower with base points near infinity has been carried out ([LS]). It does however seem that this choice of base points corresponds to a simpler version of the Teichmüller groupoids than Grothendieck's. In the Esquisse, Grothendieck suggests using base points on the moduli space corresponding to Riemann surfaces having non-generic automorphism group as well: "ceux-ci s'offrent tout naturellement, comme les courbes algébriques complexes du type $(g, n)$ envisagé, qui ont un groupe d'automorphismes (nécessairement fini) plus grand que dans le cas générique"[8].

Drinfel'd's startling contribution to the development of Grothendieck's ideas comes as a suggested application of his work on quasi-Hopf algebras. In this work, also in the article [D], he defines a group $\widehat{GT}$ (for Grothendieck-