AN ARMY OF COHOMOLOGY AGAINST RESIDUAL FINITENESS

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1. Phylogeny

In [10] we demonstrate the existence of a group of shape $\mathbb{Z}^3.\mathcal{SL}_3(\mathbb{Z})$ that fails to be residually finite. This is a locally graded group in which the normal closure of every finite set is finitely generated. H. Smith had shown that the finite residual of a group with these properties is abelian; and that if such a group is locally solvable, then its finite residual is central. He sought an instance where the finite residual is noncentral. Groups of the above shape– which are perforce perfect– induce nearly the full automorphism group on the finite residual.

Unfortunately, [10] does not provide the desired example explicitly. Rather, it offers cohomological reconnaissance of residual finiteness for arbitrary extensions, and then rolls out a potent cohomological arsenal in order to attack groups of shape $\mathbb{Z}^d.\mathcal{SL}_d(\mathbb{Z})$. It is disappointing not to have secured a concrete example after such a high-powered assault, but the battle lines are drawn rather neatly:

$$\text{residual finiteness} \iff \text{linearity} \iff \text{cohomological torsion}$$

The proof of this triple equivalence calls up Lubotzky's Criterion (see [13]) and succeeds for arithmetic subgroups of essentially all Chevalley groups (see [11]).

Cohomology enters the fray since split extensions of residually finite groups are residually finite; as are virtually split extensions; as are residually virtually split extensions. In fact, the converse is true: residually finite extensions of residually finite groups must be residually virtually split. As was known to Schur, the second cohomology group controls splitting.

Not available to Schur were the generic cohomology of algebraic groups (see [8, 9, 12]), which enables application of Lubotzky's Criterion, by first constraining the residual splitting of a 2-class to finitely many primes; nor Soulé's explicit reduction theory for $\mathcal{SL}_3(\mathbb{Z})$ (see [21]), which displays nontorsion 2-cohomology for $\mathcal{SL}_3(\mathbb{Z})$ on $\mathbb{Z}^3$; nor Borel's stable real cohomology (see [3]), which shows that $\mathcal{SL}_3(\mathbb{Z})$ is the exception, not the rule: $\mathrm{H}^2(\mathcal{SL}_d(\mathbb{Z}), \mathbb{Z}^d)$ is finite when $d \geq 7$.

We exhibit an explicit presentation for an anomalous extension $\mathbb{Z}^3.\mathcal{SL}_3(\mathbb{Z})$, and then explore a geometric representation– one we hope will prove to be flexible enough to reflect back on some of the cohomology. Ultimately we hope to provide group-theoretic rationale for Borel's weight-theoretic hypotheses.

2. Ontogeny

The proper context in which to study extensions of a given shape $M.\Gamma$ is that of an outer action, $\omega: \Gamma \to \mathcal{O}ut(M)$. To simplify matters a bit, we suppose that all groups under discussion are finitely generated. Suppose that Γ and M are residually finite. If there is a split extension in this context– *i.e.*, if Γ acts on M– then it is residually finite. In fact, split extensions are the basic building block out of which all residually finite extensions are constructed. This is because, on the one hand, virtually residually finite groups are residually finite, as are residually residually finite groups. Hence, residually virtually split extensions are residually finite. The converse is just as straightforward.

Incidentally, this theorem is well-known in the world of Galois cohomology, at least when M is finite and abelian (see [17, p. 100]). However, the example in [7] shows that the general case cannot be deduced from this special one, at least not merely by taking limits.

To see that the introduction of cohomology is not entirely frivolous, note a useful corollary of the above characterization. Suppose that M is free abelian, and that there are (up to equivalence) only finitely many extensions of shape $M.\Gamma$. If E is any one of these extensions, then its cohomology class has finite order, whence E is residually finite (see [10]). So, if we set $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$, then we find the non-residually-finite extensions of shape $M.\Gamma$ among the nonzero elements of $\mathrm{H}^2(\Gamma, M_{\mathbb{Q}})$.

Now let $\Gamma = \mathcal{SL}_d(\mathbb{Z})$, and $M = \mathbb{Z}^d$. The ideas above show that there is something distinctive about the case when $d = 3$. To begin with, if $d = 2$ then Γ is virtually free, and so every extension $M.\Gamma$ is residually finite. When d is even, $-I \in \Gamma$. Since $-I$ acts freely on M, a Frattini argument shows that $\mathrm{H}^2(\Gamma, M_{\mathbb{Q}}) = 0$ (see [10]). Finally, Borel's results (see [3]) can be applied to show that $\mathrm{H}^2(\Gamma, M_{\mathbb{Q}}) = 0$ when $d \geq 7$. (I am at a loss when $d = 5$.)

For groups and modules of the sort we are considering– arithmetic subgroups of algebraic groups and most of their simple rational modules– one can show that every nontorsion element of $\mathrm{H}^2(\Gamma, M)$ determines a non-residually-finite extension. Owing to the above characterization, this is a consequence of the following: for each prime p and each finite-index subgroup Δ, the restriction map $\mathrm{H}^2(\Gamma, M/p) \to \mathrm{H}^2(\Delta, M/p)$ is faithful. As we may assume that $\Delta = \Gamma(n)$, for some n (see [1, 14][1]), the spectral sequence of the inclusion

[1]Matsumoto's result is applied incorrectly in [11], in the case of the ring of integers in a totally imaginary number field K. In this case the congruence kernel is isomorphic with the group μ_K of roots of unity. The hole is patched simply by considering only those

$\Delta < \Gamma$ reduces this latter statement to the computation of $\mathrm{H}^k(\Gamma/p, X)$, where $\Gamma/p = \mathcal{SL}_3(\mathbb{F}_p)$; $k \leq 2$; and X is a 'smallish' module. To convey the idea behind 'small', we remark that the largest such X we need to consider is $\mathrm{Hom}(\otimes^2\mathfrak{L}, M/p)$, where $\mathfrak{L}$ is the p-modular Lie algebra.

The Classification era taught us that sporadic cohomology is most often linked to sporadic simple groups. For example, $\mathrm{Hom}_{\Gamma/p}(\otimes^2\mathfrak{L}, M/p) = 0$ except when $p = 2$ and $d = 3$. The exception gives rise to a curious 2-local parabolic in Rudvalis' simple group. We don't need the Classification, except philosophically, but it does hint strongly that the kernel of the restriction map is 0, usually.

This we can verify, using the 'generic cohomology' of finite algebraic groups (see [8, 9, 12, 22]; or, for direct computation in the case of $\mathcal{SL}_d(\mathbb{Z})$, see [2]). We find that restriction is faithful whenever $p > 5$. Because of the Pigeon Hole Principle, we deduce as a corollary that a residually finite extension E of shape $M.\Gamma$ and of infinite cohomological order, must contain a normal series $E_1 > E_2 > E_3 > \cdots$ whose sections E_k/E_{k+1} are boundedly finite p-groups, where $p = 2$, 3, or 5.

Now Lubotzky's Criterion comes to the rescue, and delivers a contradiction. Indeed, an extension containing such a series would be linear, and hence contain such a series for almost every prime. This is in conflict with the cohomological calculations, and so E fails to be residually finite.

Now we are left with the task of finding nontorsion 2-cohomology. It turns out that there is a 'reduction theory', due to Soulé, which gives the cohomology of $\mathcal{SL}_3(\mathbb{Z})$ about as explicitly as one is likely to see (see [21]). It works best when 6 is invertible on the coefficient module, but we can arrange this in our case, by extending scalars to $\mathbb{Q}$. One computes that $\mathrm{H}^k(\Gamma, M_\mathbb{Q}) = 0$ whenever $k \neq 2$, deducing that the Euler characteristic of $M_\mathbb{Q}$ is equal to the dimension of $\mathrm{H}^2(\Gamma, M_\mathbb{Q})$. On the other hand, the Euler characteristic can be computed at the chain level, using Soulé's reduction theory: The answer is 1.

3. Presentation

Armed with the answer, let us try to hunt down these elusive extensions exactly. As before, set $\Gamma = \mathcal{SL}_3(\mathbb{Z})$ and $M = \mathbb{Z}^3$. We will show that the following presentation— based on the Steinberg relations (see [22]) and the action of Γ on M— defines an extension that fails to be residually finite.

$$E = \left\langle x_{ij}, e_k \ (1 \leq i,j,k \leq 3;\ i \neq j) \ \middle|\ [e_k, x_{ij}] = \begin{cases} e_j, & \text{if } k = i; \\ 1, & \text{if } k \neq i; \end{cases} \right.$$
$$\left. [e_k, e_{k'}] = 1; \quad [x_{ij}, x_{kj}] = 1; \quad [x_{ij}, x_{jk}] = e_j^2 e_k, \text{ if } k \neq i. \right\rangle$$

primes that do not divide $|\mu_K|$, and then applying the Transfer one more time.

We do this by finding an explicit 2-cocycle for the extension. I should note, however, that I do not yet know how to prove that this extension fails to be residually finite, except to invoke the results from [10].

Since $\mathcal{GL}_2(\mathbb{Z})$ is virtually free, some Baer power of E is generated by copies of the root subgroups Λ_1 and Λ_2, which intersect in the diagonal subgroup Δ:

$$\Pi_1 = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$\Delta = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix} \qquad \Lambda_2 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Now any Baer power of a non-residually-finite extension of this shape must itself fail to be residually finite, since any multiple of a cocycle of infinite order must itself have infinite order. So, without loss of generality, our extension is generated by the Λ_i.

However, if E fails to be residually finite, then it cannot be generated by copies of the parabolic overgroups Π_i (see [15, 16]– more on this anon). This fact allows us to find the cocycle explicitly, by restricting to the Π_i. In fact, since every pair of root elements x_{ij} lie together in a conjugate of Π_2, it suffices to consider the cocycle on this subgroup.

Since Π_2 is simply a split extension $U_2.\Lambda_2$, where $U_2 \cong \mathbb{Z}^2$, the neatest description of the cohomological character of the parabolics in such extensions would be given by the characteristic classes of Charlap and Vasquez (see [5, 6]), which were explored by Sah (see [18]). However, the cohomology is easy to compute directly.

To do so, we apply the spectral sequence. So, first we compute $\mathrm{H}^*(U_2, M_\mathbb{Q})$, by means of the cohomology sequence associated to the decomposition

$$0 \to [M_\mathbb{Q}, U_2] \to M_\mathbb{Q} \to M_\mathbb{Q}/[M_\mathbb{Q}, U_2] \to 0.$$

The induced long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(U_2,[M_\mathbb{Q},U_2]) &\xrightarrow{\cong} \mathrm{H}^0(U_2,M_\mathbb{Q}) \xrightarrow{0} \mathrm{H}^0(U_2,M_\mathbb{Q}/[M_\mathbb{Q},U_2]) \xrightarrow{\cong} \\ \mathrm{H}^1(U_2,[M_\mathbb{Q},U_2]) &\xrightarrow{0} \mathrm{H}^1(U_2,M_\mathbb{Q}) \longrightarrow \mathrm{H}^1(U_2,M_\mathbb{Q}/[M_\mathbb{Q},U_2]) \longrightarrow \\ \mathrm{H}^2(U_2,[M_\mathbb{Q},U_2]) &\xrightarrow{0} \mathrm{H}^2(U_2,M_\mathbb{Q}) \xrightarrow{\cong} \mathrm{H}^2(U_2,M_\mathbb{Q}/[M_\mathbb{Q},U_2]) \longrightarrow 0. \end{aligned}$$

is Λ_2-equivariant, and we find that

$$\mathrm{H}^1(U_2, M_\mathbb{Q}) \cong S_2(\mathbb{Q}) = \{2 \times 2 \text{ symmetric matrices}\}; \quad \mathrm{H}^2(U_2, M_\mathbb{Q}) \cong \mathbb{Q}^2.$$

Indeed, a matrix $\delta: \mathbb{Z}^2 \to \mathbb{Q}^2$ lifts to a 1-cocycle $\tilde{\delta}: \mathbb{Z}^2 \to M_\mathbb{Q}$ precisely when there is a function $q: \mathbb{Z}^2 \to \mathbb{Q}$ satisfying the following identity:

$$q(v + w) = q(v) + q(w) + \delta(v) \cdot w,$$

where $\cdot$ denotes inner product. This means that δ is a symmetric matrix, and q is its associated quadratic form: $q(v) = \frac{1}{2}\delta(v) \cdot v$. In particular, the cocycle restricts to 0 on U_2.

Moreover, Λ_2 acts through similarity on $\mathrm{H}^1(U_2, M_{\mathbb{Q}})$; and through multiplication by the determinant on $\mathrm{H}^2(U_2, M_{\mathbb{Q}})$. Again, Λ_2 is virtually free, and so the spectral sequence tells us that

$$\mathrm{H}^2(\Pi_2, M_{\mathbb{Q}}) = \mathrm{H}^1(\mathcal{GL}_2(\mathbb{Z}), S_2(\mathbb{Q})).$$

It remains, then, to identify $\mathrm{H}^1(\mathcal{GL}_2(\mathbb{Z}), S_2(\mathbb{Q}))$.

We apply the Mayer-Vietoris sequence ([4, p. 37]; but compare with [23]), since $\mathcal{SL}_2(\mathbb{Z})$ is simply a freely amalgamated product: $\mathcal{SL}_2(\mathbb{Z}) = \langle s \rangle *_{\pm I} \langle u \rangle$, where $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $u = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. Thus,

$$\begin{aligned} 0 \longrightarrow\ & S_2(\mathbb{Q})^s \oplus S_2(\mathbb{Q})^u \longrightarrow S_2(\mathbb{Q}) \xrightarrow{p} \mathrm{H}^1(\mathcal{SL}_2(\mathbb{Z}), S_2(\mathbb{Q})) \\ \xrightarrow{q}\ & \mathrm{H}^1(\langle s \rangle, S_2(\mathbb{Q})) \oplus \mathrm{H}^1(\langle u \rangle, S_2(\mathbb{Q})) \longrightarrow 0. \end{aligned}$$

The sum of the fixed-point subspaces– which are indicated by the superscripts– is 2-dimensional. Hence the image of p is 1-dimensional, and $\mathrm{H}^1(\mathcal{SL}_2(\mathbb{Z}), S_2(\mathbb{Q}))$ is 5-dimensional.

If $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\mathrm{H}^1(\mathcal{GL}_2(\mathbb{Z}), S_2(\mathbb{Q})) \cong \mathrm{H}^1(\mathcal{SL}_2(\mathbb{Z}), S_2(\mathbb{Q}))^H$. Since the Mayer-Vietoris sequence is equivariant for H, we find that

$$\mathrm{H}^1(\mathcal{SL}_2(\mathbb{Z}), S_2(\mathbb{Q}))^H = \left\{ s \mapsto \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix};\ u \mapsto \begin{bmatrix} b & b \\ b & 0 \end{bmatrix} \right\}$$

Indeed, H is nontrivial on $S_2(\mathbb{Q})/(S_2(\mathbb{Q})^s + S_2(\mathbb{Q})^u)$; and the map q is given by composing differentials $\pm s \mapsto \begin{bmatrix} c & a \\ a & -c \end{bmatrix}$ and $\pm u \mapsto \begin{bmatrix} b & b+d \\ b+d & d \end{bmatrix}$, with surjections $\mathcal{SL}_2(\mathbb{Z}) \to \langle s \rangle / \pm I$ and $\mathcal{SL}_2(\mathbb{Z}) \to \langle u \rangle / \pm I$, respectively.

Let us translate these 1-cocycles for Λ_2 into 2-cocycles for Π_2. Write the elements of $\mathrm{H}^1(\Lambda_2, S_2(\mathbb{Q}))$ as $x \mapsto \delta_x$, where $\delta_x \in S_2(\mathbb{Q})$. So, we can regard each δ_x as a homomorphism from U_2 to $\mathbb{Q}^2$. For each x in Λ_2, let $q_x: U_2 \to \mathbb{Q}$ be the quadratic form associated to δ_x; this lifts δ_x to a differential $\tilde{\delta}_x: U_2 \to M_{\mathbb{Q}}$:

$$\tilde{\delta}_x(u) = (q_x(u), \delta_x(u)).$$

The formula for the associated 2-cocycle φ is now easy to write down:

$$\varphi(xv, yw) = \tilde{\delta}_y(v^y) \cdot w.$$

To finish, we need to determine which of these 2-cocycles are Γ-invariant. This is quite straightforward. Set $t^+ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $t^- = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have that $t^+ = s^{-1}u$ and $t^- = su^{-1}$, whence

$$\delta_{t^+} = \begin{bmatrix} b-2a & b-a \\ b-a & 0 \end{bmatrix}; \qquad \delta_{t^-} = \begin{bmatrix} 0 & b-a \\ b-a & b-2a \end{bmatrix}.$$

The above formula tells us that $\varphi(x_{21}, x_{23}) = 0$. But the pair x_{21}, x_{23} is conjugate in Γ to the pair $x_{23}, x_{21}{}^{-1}$, which tells us that $a = b$ if φ is to be invariant. If we choose $a = -2$, then $\varphi(x_{31}, x_{23}) = (1, 0, 2)$. From this value we can obtain the above presentation, by means of conjugation in Γ.

I have done the above calculations in absolutely excruciating detail in order to remove any further need for cohomology in determining E ‘explicitly’. However, having E in hand brings me no closer to establishing its residual finiteness, unless I revert to cohomology. In the next section, I outline a geometric representation of E that I hope will shed new light.

4. Representation

Let $\mathfrak{K}$ be a ‘nice’ topological space admitting the group Γ. If π' is a Γ-invariant normal subgroup of $\pi_1(\mathfrak{K})$, then the elements of Γ lift to automorphisms of the cover $\mathfrak{G}$ corresponding to π', by the Homotopy Lifting Property. The ambiguity of each lift is the group of covering transformations, $\pi_1(\mathfrak{K})/\pi'$. In particular, for each Γ-module M we have a map $\mathrm{Hom}_\Gamma(\mathrm{H}_1(\mathfrak{K}), M) \to \mathrm{H}^2(\Gamma, M)$.

Let us specialize to the situation at hand: $\Gamma = \mathcal{SL}_3(\mathbb{Z})$, $M = \mathbb{Z}^3$, and E is an extension that is split over the root subgroups Λ_i. The free product can be viewed as automorphisms of a tree $\mathfrak{B}$, whose vertices are the Γ-conjugates of the Λ_i (see [19]). Thus, any completion of this amalgam embeds as automorphisms of the corresponding quotient of $\mathfrak{B}$:

$$\begin{array}{ccccccc} \Lambda_1 *_\Delta \Lambda_2 = & F & \to & E & \to & \Gamma \\ & \cup & & \cup & & \\ \pi_1(\mathfrak{K}) = & \pi & \to & M & & \\ & \cup & & & & \\ & \pi' & & & & \end{array} \qquad \begin{array}{ccccc} \mathfrak{B} & \to & \mathfrak{G} & \to & \mathfrak{K} \\ & & \| & & \| \\ & & \mathfrak{B}/\pi' & & \mathfrak{B}/\pi \\ & & \multicolumn{3}{c}{\underbrace{\qquad\qquad\quad}_{M}} \end{array}$$

Note that $\mathfrak{K}$ embeds in the graph $\mathfrak{K}_\mathbb{Q}$ on point-line antiflags in $\mathbb{P}(M_\mathbb{Q})$, where a pair of antiflags determine an edge when their geodesic closure in the Tits building is an apartment. Alternatively, the graph $\mathfrak{K}_\mathbb{Q}$ is simply the commuting graph on reflections in $\mathcal{GL}_3(\mathbb{Q})$, and was studied by A. Wagner (see [24]).

Since $\mathrm{H}_1(\mathfrak{K})$ is the abelianization of the covering transformations π, we wish to find an explicit homomorphism $\mathrm{H}_1(\mathfrak{K}_\mathbb{Q}) \to M$ that is nontrivial on $\mathrm{H}_1(\mathfrak{K})$. Each such homomorphism defines an extension E of the right shape.

Note that if E were generated by the Π_i, then we obtain an analogous graph, which is simply be the Tits building of $\mathbb{P}(\mathbb{Q}^3)$. (Since $\mathbb{Z}$ is a 'Bezout ring', Γ is flag-transitive on the building, and so it does not matter whether we work over $\mathbb{Z}$ or $\mathbb{Q}$.) Its homology is the Steinberg module, and there are no nonzero homomorphisms from the Steinberg module to M (see [15, 16]), whence E would be split. This justifies our restriction φ to the Π_i, in the previous section.

Let us return to $\mathfrak{K}_{\mathbb{Q}}$. Now $\Gamma_{\mathbb{Q}}$ has rank 8 on antiflags: a basepoint $*$; a single suborbit at distance 1; 5 suborbits at distance 2, of which 3 contain points joined uniquely to $*$ by a geodesic; and a single suborbit at distance 3. If $* = p, L$, and p', L' is a second antiflag, then we label p', L' according to the following scheme:

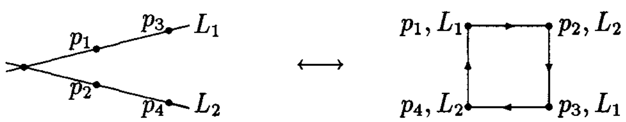
$*:\ p' = p;\ L' = L$	$\delta:\ p' \notin L;\ p \notin L';\ l \cap L' \notin p + p'$
$\alpha:\ p' \in L - \{p\};\ p \in L' - \{p'\}$	$\epsilon:\ p' = p;\ L' \neq L$
$\beta:\ p' \notin L;\ p \in L' - \{p'\}$	$\zeta:\ p' \neq p;\ L' = L$
$\gamma:\ p' \in L - \{p\};\ p \notin L'$	$\eta:\ p' \notin L;\ p \notin L';\ L \cap L' \in p + p'$

So, there are two types of quads: $* \to \alpha \to \epsilon \to \alpha \to *$ and $* \to \alpha \to \zeta \to \alpha \to *$.

Following a suggestion of D. Evans, we view the oriented edges in $\mathfrak{K}_{\mathbb{Q}}$ as corresponding naturally to ordered frames. Thus, $\mathrm{H}_1(\mathfrak{K}_{\mathbb{Q}})$ is a homomorphic image of the permutation module Φ on frames $p.q.r$. The boundaries $\mathrm{B}_1(\mathfrak{K}_{\mathbb{Q}})$ are sums of terms of the form $p.q.r + r.q.p$, and so we may replace Φ by the module induced from the sign character of $\mathfrak{S}_3$.

An oriented triangle is represented in Φ as $p.q.r + q.r.p + r.p.q$, which is left invariant by a subgroup isomorphic to $\mathfrak{A}_4$. Since this subgroup leaves invariant no nonzero vector in M, we deduce that the portion of the homology generated by the triangles maps to 0 under the homomorphism we seek.

An oriented quad



is represented in Φ as $p_1.x.p_2 + p_2.x.p_3 + p_3.x.p_4 + p_4.x.p_1$, where $x = L_1 \cap L_2$. Let this quad be denoted $\phi(x; p_1, p_2, p_3, p_4)$. On the other hand, an oriented quad of dual type is easier to describe through a dual description of the group of chains. So, an ordered triple of noncollinear points $p.p'.p''$ corresponds to an ordered triple of noncoïncident lines $L.L'.L''$, where $p = L' \cap L''$, $p' = L'' \cap L$, and $p'' = L \cap L'$. Thus, the oriented quad of dual type correspond to sums of the form $L_1.X.L_2 + L_2.X.L_3 + L_3.X.L_4 + L_4.X.L_1$, where $X = p_1 + p_2$.

The portion of Φ generated by the first type of quad is induced from the span of $\{\phi(x; p_1, p_2, p_3, p_4) | p_i \neq x\}$, considered as a module for the stabilizer of a point Γ_x. Let the span of the quads and triangles be denoted Θ; and let the span of the quads based at x be denoted Φ_x. Choose a generator v for the projective point x. Take a basis of Φ_x consisting of quads, and map any part of this basis to v, the rest to 0. This gives a Γ_x-equivariant map $\Phi_x \to M$, from which we can induce to a Γ-equivariant map $\Theta \to M$. (Recall that the triangles map to 0.) These maps span all equivariant maps $\Theta \to M$, so if all yield residually finite extensions, then the explicit homomorphism we seek is defined modulo the triangles and quads.

This idea runs out of gas at the pentagons, since these have trivial centralizers. One needs to address the pentagons, since one can show that the triangles and quads generate a proper submodule. On the other hand, the pentagons and triangles *do* generate all of the homology. Moreover, there does not appear to be any invariant complement to Θ in $\mathrm{H}_1(\mathfrak{K}_\mathbb{Q})$.

I expect that the above homomorphisms extend to all of the homology, but even so I have as yet no means of deciding whether they result in residually finite extensions, other than to verify that the 2-cocycles they yield have infinite order. Nevertheless, this approach seems promising, and appears to be tailor-made for the low-rank groups that Borel's theory does not cover.

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ON SOME QUESTIONS CONCERNING SUBNORMALLY MONOMIAL GROUPS

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In his papers [4], [5] and [6] Guan-Aun How described some properties of SM (subnormally monomial) groups. He proved that the class of SM-groups is the intersection of the class of CSF (chiefly sub-Frobenius) and the class X of those solvable groups G, for which for all primes p and for all subgroups A, $O^p(A)$ has no central p-factor. Among other things, he proved that the class of SM-groups is closed under taking direct products, factor groups and subgroups. First we consider the relation between SM-groups, subgroup-closed M-groups and supersolvable groups, showing that these classes are all distinct. For the class of Frobenius groups the first two classes coincide, and they properly contain the class of supersolvable Frobenius groups. For Frobenius complements the three classes are equal. The class SM is not closed under extensions. We show that even the split extension of an abelian group with an SM-group can be non-SM. On the basis of the notion of relative M-groups, we introduce the notion of relative SM-groups. We investigate whether some results which are known to be true for relative M-groups, remain true for relative SM-groups or not. Some of them remain true: we show that every SM-group is relative SM with respect to every abelian normal subgroup. According to [7], if G/N is supersolvable, then G is relative M-group with respect to N. The analogous statement is not true for SM: it may happen that G/N is supersolvable, but G is not relative SM with respect to N. However, if G/N is nilpotent, then G is relative SM with respect to N. Every Frobenius SM-group is relative SM with respect to its kernel, moreover, this property characterizes Frobenius SM-groups. We also show that a Frobenius group is SM if and only if its complement is SM. So fixed point free extension with an SM-group is always SM. Finally we show an algorithm which tests whether a group is SM or not. This algorithm can be implemented in GAP, it has been tested on some groups already (e.g. on Examples 1.5 and 1.6). It is of polynomial complexity as a function of the order of the group, provided that the multiplication table of the group is known. Throughout this paper all groups are finite and all characters are complex-valued.

1.

Definition 1.1. An irreducible character of a group G is called *monomial*