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1

FUNDAMENTAL PROPERTIES OF VORTICITY

1.1 Relation between velocity and vorticity

The motion of a fluid is described by a vector field $\mathbf{u}(\mathbf{x}, t)$.[1] The curl of the velocity is called the vorticity $\boldsymbol{\omega}(\mathbf{x}, t)$. In the various notations

$$\boldsymbol{\omega}(\mathbf{x}, t) = \omega_i \equiv \operatorname{curl} \mathbf{u} = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = (\xi, \eta, \zeta). \quad (1)$$

It follows from the definition that the vorticity is solenoidal, that is,

$$\operatorname{div} \boldsymbol{\omega} = \frac{\partial \omega_i}{\partial x_i} = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0. \quad (2)$$

The significance and importance of vorticity for the description and understanding of fluid flow stems from the facts, first, that (1) may be inverted to give the velocity field as an integral over the vorticity field, and, second, that when the viscous diffusion of vorticity is negligible, the fluid is barotropic (i.e., the density ρ is a single-valued function of the pressure p) and the external forces are conservative, then the vorticity satisfies conservation principles known as the Helmholtz laws (see §5), which allow the vorticity to be 'followed'. In particular, vorticity is not created and a compact distribution of vorticity remains compact,[2] so that the structure

[1] It is convenient to use various notations. The velocity will be denoted by a vector $\mathbf{u}$, or by Cartesian components (u, v, w), or by tensor components (u_1, u_2, u_3), or by unit vector decomposition $u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, according to circumstances. The position vector $\mathbf{x}$ will be denoted by (x, y, z), or $\mathbf{x} = (x_1, x_2, x_3)$ or $\mathbf{r} = (r_1, r_2, r_3)$, and similarly for other vectors. The summation convention will be implied when tensor notation is used.

[2] We do not use 'compact' in its pure mathematical sense, but in the ordinary sense of being localised to a finite region, outside of which the vorticity is zero or at most exponentially small.

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and evolution of the fluid flow is more economically described in terms of the vorticity field than in terms of the velocity field.

We need to know conditions for which the inversion is possible and determines $\mathbf{u}$ uniquely. Sufficient but not necessary are the following six statements:

(i) The velocity field is solenoidal, that is,

$$\operatorname{div} \mathbf{u} = 0. \tag{3}$$

(This condition is satisfied if the fluid is incompressible, that is, the material derivative $D\rho/Dt \equiv \partial\rho/\partial t + \mathbf{u}\cdot\nabla\rho = 0$; it is not necessary that the fluid be homogeneous, that is, ρ =constant.)

(ii) The region occupied by the fluid is singly connected.

(iii) The normal component of fluid velocity, U_n, is given on all bounding surfaces S.

(iv) The velocity vanishes at infinity when the fluid is unbounded.

(v) The normal component of vorticity vanishes on S.

(vi) The vorticity field is compact.

The velocity is then given uniquely by the sum of a solenoidal vector potential component and an irrotational scalar component

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_v(\mathbf{x}, t) + \nabla\Phi \tag{4}$$

where

$$\mathbf{u}_v = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\mathbf{x}', t)\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,d\mathbf{x}' = -\frac{1}{4\pi}\int \boldsymbol{\omega}(\mathbf{x}', t)\times\nabla\frac{1}{|\mathbf{x}'-\mathbf{x}|}\,d\mathbf{x}', \tag{5}$$

and $\nabla\Phi$ is determined uniquely (e.g., see Lamb [1932 §35]) by the classical potential problem

$$\nabla^2\Phi = 0, \quad \frac{\partial\Phi}{\partial n} = U_n - \mathbf{u}_v\cdot\mathbf{n} \quad \text{on } S, \quad \Phi\to 0 \text{ as } \mathbf{x}\to\infty. \tag{6}$$

In (4), $\mathbf{u}_v$ is a solenoidal field satisfying

$$\operatorname{curl} \mathbf{u}_v = \boldsymbol{\omega}, \tag{7}$$

and $\nabla\Phi$ is the irrotational, solenoidal field which can be added to satisfy a single boundary condition on the velocity field on S. If there are no bounding surfaces, $\Phi = 0$.

The proof of (4) is a standard exercise in vector analysis. A brief heuristic derivation is given here in order to demonstrate how the six conditions enter. For brevity, we write $\boldsymbol{\omega}' = \boldsymbol{\omega}(\mathbf{x}', t)$, $r = |\mathbf{x}-\mathbf{x}'|$, $\partial/\partial\mathbf{x} = \nabla$,

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$\partial/\partial\mathbf{x}' = \nabla'$, and so on. The following two vector identities are used repeatedly;

$$\operatorname{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\operatorname{curl}\mathbf{a} - \mathbf{a}\cdot\operatorname{curl}\mathbf{b}, \tag{8}$$

$$\operatorname{curl}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}.\nabla)\mathbf{a}. \tag{9}$$

The following results of generalized function theory are also applied:

$$\operatorname{curl}\operatorname{grad}(1/r) = 0, \tag{10}$$

$$\nabla^2(1/r) = -4\pi\,\delta(\mathbf{x}-\mathbf{x}'). \tag{11}$$

First we consider $\mathbf{u}_v$. Taking the divergence of (5) and using (8) and (10), we have

$$\operatorname{div}\mathbf{u}_v = -\frac{1}{4\pi}\int \operatorname{div}(\boldsymbol{\omega}'\times\nabla(1/r))\,d\mathbf{x}' = \frac{1}{4\pi}\int \boldsymbol{\omega}'\cdot\operatorname{curl}\operatorname{grad}(1/r)\,d\mathbf{x}' = 0. \tag{12}$$

Taking the curl of (5), and applying (9), we have

$$\begin{aligned}\operatorname{curl}\mathbf{u}_v &= -1/4\pi\int \boldsymbol{\omega}'\nabla^2(1/r)\,d\mathbf{x}' + 1/4\pi\int(\boldsymbol{\omega}'\cdot\nabla)\nabla(1/r)\,d\mathbf{x}' \\ &= \boldsymbol{\omega} - 1/4\pi\int(\boldsymbol{\omega}'\cdot\nabla')\nabla(1/r)\,d\mathbf{x}',\end{aligned} \tag{13}$$

on using (11), the property of the delta function ($\int f(x)\delta(\mathbf{x})d\mathbf{x} = f(0)$), and $\nabla(1/r) = -\nabla'(1/r)$. Since $\operatorname{div}'\boldsymbol{\omega}' = 0$, we can rewrite the last term in (13) as

$$(\boldsymbol{\omega}\cdot\nabla')\nabla(1/r) = \nabla'\cdot(\boldsymbol{\omega}'\nabla(1/r)),$$

and then from the divergence theorem,[3]

$$\int(\boldsymbol{\omega}'\cdot\nabla')\nabla\left(\frac{1}{r}\right)d\mathbf{x}' = -\int(\boldsymbol{\omega}'\cdot\mathbf{n})\nabla\left(\frac{1}{r}\right)dS' = 0 \tag{14}$$

by virtue of conditions (v) and (vi). Hence (7) follows. Conditions (i), (ii), (iii) and (iv) then specify a unique solenoidal irrotational velocity field $\nabla\Phi$ given by (6), which ensures that $\mathbf{u}$ is uniquely determined by $\boldsymbol{\omega}$. Note that only one boundary condition on $\mathbf{u}$ may be applied on S. Thus, in general, a no-slip boundary condition cannot be applied.

An alternative procedure employs the vector potential $\mathbf{A}(\mathbf{x},t)$, which exists when (3) is satisfied, defined by

$$\mathbf{u}_v = \operatorname{curl}\mathbf{A}. \tag{15}$$

[3] We shall apply the convention that $\mathbf{n}$ is the unit normal vector on a bounding surface S directed out of the fluid.

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To determine **A**, take the curl of (15), giving

$$\boldsymbol{\omega} = \operatorname{curl}\operatorname{curl}\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \nabla^2\mathbf{A}. \tag{16}$$

The vector potential is arbitrary to the addition of the gradient of a scalar. Choose the scalar (i.e., fix the gauge of **A**) by requiring that $\operatorname{div}\mathbf{A} = 0$. Then (16) is the Poisson equation

$$\nabla^2\mathbf{A} = -\boldsymbol{\omega}, \tag{17}$$

with solution

$$\mathbf{A} = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}'}{r}\,d\mathbf{x}'. \tag{18}$$

It is now necessary to verify that this **A** is solenoidal. By arguments similar to those employed above (see also Batchelor[1967 §2.4.]), we find that

$$\operatorname{div}\mathbf{A} = \frac{1}{4\pi}\int \boldsymbol{\omega}'.\nabla(\frac{1}{r})\,d\mathbf{x}' = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}'\cdot\mathbf{n}'}{r}\,dS' = 0,$$

when conditions (v) and (vi) are satisfied. Taking the curl of (18), we obtain (5). An alternative form is

$$\mathbf{u}_v = \frac{1}{4\pi}\int \frac{1}{r}\operatorname{curl}\boldsymbol{\omega}'\,dV' - \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}'\times\mathbf{n}'}{r}\,dS'. \tag{19}$$

Instead of using the Green's function (18), the inversion can be carried out using only solutions of the scalar Laplace and Poisson equations with boundary conditions on S. There is no unique recipe, and the best procedure depends upon the application. For example, take **B** as any solution of the three scalar equations for its Cartesian co-ordinates,

$$\nabla^2\mathbf{B} = -\boldsymbol{\omega}. \tag{20}$$

Note that $\operatorname{div}\mathbf{B}$ is harmonic because of (2), but not necessarily zero. Let g denote the value of $\operatorname{div}\mathbf{B}$ on S. Now solve the equation for the Cartesian components of **C**, defined by

$$\nabla^2\mathbf{C} = 0, \quad \operatorname{div}\mathbf{C} = -g \quad \text{on } S. \tag{21}$$

This equation is also underdetermined and can be solved in an infinity of ways. For example, one can take $C_2 = C_3 = 0$, and $\partial C_1/\partial x = -g$ on S. Let f denote the solution of

$$\nabla^2 f = 0, \quad f = -g \quad \text{on } S. \tag{22}$$

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Then

$$C_1 = \int_0^x f(\xi, y, z)d\xi + h(y, z), \quad \nabla^2 h = -\frac{\partial f}{\partial x}(0, y, z) \tag{23}$$

is a solution of (21), since $\partial C_1/\partial x = f$ and $\nabla^2 C_1 = \nabla^2 h + \partial f(0, y, z)/\partial x$.

Regardless of how **B** and **C** are determined in detail, $\mathbf{A} = \mathbf{B}+\mathbf{C}$ satisfies (17). Further, div **A** is harmonic and vanishes on S by construction. Hence, $\operatorname{div}\mathbf{A} = 0$ inside S, and **A** is a vector potential.

As a simple example, consider the case when $\boldsymbol{\omega} = (1, 0, 0)$ inside a sphere of unit radius. Then

$$\mathbf{B} = -\frac{x^2 + y^2 + z^2}{6}\mathbf{i}$$

is a solution of (20), where **i** is the unit vector in the x-direction. Then $g = -x/3$, and $f = x/3$ is the solution of (22). We find from (23) that $h = -\frac{1}{12}(y^2 + z^2)$, and

$$\mathbf{C} = \left(\frac{x^2}{6} - \frac{1}{12}(y^2 + z^2)\right)\mathbf{i}.$$

Thus

$$\mathbf{A} = -\frac{1}{4}(y^2 + z^2)\mathbf{i}$$

is solenoidal and satisfies (17). The corresponding velocity field is

$$\mathbf{u}_v = (0, -\tfrac{1}{2}z, \tfrac{1}{2}y).$$

Hirasaki and Hellums [1970] consider a three-dimensional rectangular geometry and use as boundary conditions on the vector potential **A** that satisfies (17)

$$\frac{\partial}{\partial n}(\mathbf{n}\cdot\mathbf{A}) = 0, \quad \mathbf{n}\times\mathbf{A} = 0, \tag{24}$$

which leads to three scalar Poisson equations with part Dirichlet and part Neumann conditions.

An arbitrary value U_n for the normal velocity is satisfied as before by adding the gradient of a harmonic function ϕ to curl **A** such that $\partial\phi/\partial n = U_n - \mathbf{n}\cdot\operatorname{curl} A$.

Note that with direct inversion, which does not use the free-space Green's function (5) or (18), it is not necessary to require that conditions (v) and/or (vi) are satisfied. We describe later (§2.4) how to extend the vorticity field so that the Green's function can be used when (v) is not satisfied.

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1.2 Vorticity and rotation

A kinematical interpretation of vorticity is provided by analysing the relative motion near a point. The relative velocity δu_i of two fluid particles separated by δx_i can be written

$$\delta u_i = \partial u_i/\partial x_j \, \delta x_j = e_{ij}\delta x_j + \Omega_{ij}\delta x_j, \tag{1}$$

where

$$e_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \tag{2}$$

and

$$\Omega_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i) = -\tfrac{1}{2}\epsilon_{ijk}\omega_k \tag{3}$$

are the rate of strain tensor and 'vorticity tensor', respectively. Note the converse relation

$$\omega_i = -\epsilon_{ijk}\Omega_{jk}. \tag{4}$$

The two contributions to the right-hand side of (1) constitute a pure straining motion and rigid body rotation. In the pure straining motion, line elements are extended or contracted, and spheres are deformed into quadrics with principal axes along those of the rate of strain tensor. In the rigid body rotation, line elements stay of constant length and spheres remain spheres while rotating with angular velocity

$$\boldsymbol{\Omega} = \tfrac{1}{2}\boldsymbol{\omega}. \tag{5}$$

Regions of fluid in which the vorticity is identically zero are said to be in irrotational motion. Solid particles do in general rotate when suspended in fluid in irrotational motion, even if the fluid is assumed to be ideal (i.e., zero viscosity). This is because a body of general shape is subject to a torque when suspended in irrotational flow of an ideal fluid (Landau and Lifshitz [1959 §11]). The particular feature of irrotational flow is that the torque vanishes for non-spherical bodies which have rotational symmetry about three perpendicular axes[4] (e.g., cubes or regular polyhedra), but does

[4] The same is true for microscopic particles in a real fluid. An irrotational solution of the Euler equation for a uniform fluid is also a solution of the Navier–Stokes equations, and bodies of the appropriate symmetry will experience no torque when sufficiently small for the motion around them to be described by the creeping flow equations. Consider, for example, a dumb-bell with one end at the origin and the other at the point (x, y) in the two-dimensional plane strain $u = \alpha x$, $v = -\alpha y$. Assuming that the force on the ends is proportional to the relative velocity of ball and fluid, there will be a torque $\propto 2\alpha xy$ tending to rotate the body. If the body consists of two equal

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not vanish in general when such bodies are suspended in rotational flow, that is, $\omega \neq 0$. In a two-dimensional irrotational flow, a square or cross will not rotate when carried along with the flow, but a rod or diagonal cross or ellipse will.

The vorticity is twice the average angular velocity around an infinitesimal circle since

$$\frac{1}{2\pi l}\oint \frac{\mathbf{u}}{l}\cdot d\mathbf{s} = \frac{1}{2\pi l^2}\int \boldsymbol{\omega}\cdot d\mathbf{A} \quad \text{by Stokes's theorem} \\ \to \tfrac{1}{2}\boldsymbol{\omega} \qquad \text{as} \quad l \to 0. \tag{6}$$

A dynamical interpretation in terms of angular momentum density follows from the calculation of the angular momentum about its centroid of an infinitesimal fluid particle. Suppose the fluid particle has volume δV and centroid at $\mathbf{x}$. Its angular momentum $\delta\mathbf{A}$ about its centroid is

$$\delta A_i = \rho\int_{\partial V} \epsilon_{ijk}\,\delta x_j\,\delta u_k\,dV. \tag{7}$$

Substituting (1) for δu_k, we obtain

$$\delta A_i = \epsilon_{ijk}\,e_{kl}\,I_{jl} + \tfrac{1}{2}(\delta_{ij}I_{kh} - I_{ij})\omega_j, \tag{8}$$

where

$$I_{ij} = \rho\int_{\partial V} \delta x_i\,\delta x_j\,dV \tag{9}$$

is the inertia tensor of the fluid particle. The first term on the right-hand side of (8) vanishes if the body has spherical symmetry, as is the case for cubes, in which case the inertia tensor is diagonal. The second term is the angular momentum of the fluid particle rotating as a solid body with angular velocity $\frac{1}{2}\boldsymbol{\omega}$. Vorticity can therefore be identified as proportional to the angular momentum of fluid particles whose inertia tensors have spherical symmetry.[5]

dumbbells joined at right angles with ends at (x, y) and (x', y'), the sum of the torques is proportional to $2\alpha(xy + x'y') = 0$.

[5] It can be speculated that a gyroscopic interpretation can be provided for vorticity and that the motion of fluids endowed with vorticity can be better understood by examining gyroscopic motion. For example, Coles [1967] pointed out that the Rayleigh criterion for the stability of flow between rotating cylinders is equivalent to the criterion that vorticity and angular rotation (precession) are in the same sense, which is the condition for stability of a spinning top.

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1.3 Circulation

A scalar functional of considerable importance in the description of vortex flows is the circulation Γ around a simple closed curve C, defined as the line integral of the velocity.

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s}. \tag{1}$$

We shall say that a curve is reducible if it can be shrunk continuously to a point without going outside the fluid. It follows from Stokes's theorem that the circulation around a reducible curve is equal to the flux of vorticity through an open surface A bounded by the curve, that is,

$$\int_A \boldsymbol{\omega} \cdot \mathbf{n}\, dS = \oint_C \mathbf{u} \cdot d\mathbf{s} = \Gamma. \tag{2}$$

Some convention is required for the sense in which the circulation is taken and the direction of the normal to the surface. We shall suppose that the relation is that described by a right-hand-threaded screw.

The vanishing of the circulation for all closed curves implies that the vorticity is zero and the flow is irrotational. The converse is true if the fluid is contained in a simply connected region, but may be false if the region is multiconnected. Irrotational flow about a torus may have non-zero circulation around curves which thread the hole.

The circulation is important for its conservation principles (Kelvin's circulation theorem §6), its relation to the forces on bodies (the Kutta lift §3.1) and its use in vortex sheet dynamics as a variable for the parametrization of the sheet shape (the Birkhoff–Rott equation §8.1).

1.4 Vortex lines and tubes

In a region of fluid where the vorticity does not vanish identically, curves drawn parallel to the vorticity vector at each point of the curve are known as vortex lines. They are the solution families of the differential equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}. \tag{1}$$

The vortex lines passing through the points of a reducible curve define a cylindrical-like volume called a vortex tube. This term commonly implies

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that the cross-sections are small and oval, or even infinitesimal. The tube has the property that the vorticity is everywhere parallel to its surface, that is, $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on the surface of a vortex tube. It follows that the flux of vorticity through any cross-section of the tube is constant, since if A_1 and A_2 are two cross-sections, the divergence theorem gives

$$\int_{A_1} \boldsymbol{\omega} \cdot \mathbf{n}\, dS - \int_{A_2} \boldsymbol{\omega} \cdot \mathbf{n}\, dS = \int_{A_1 - A_2} \operatorname{div} \boldsymbol{\omega}\, dV = 0. \qquad (2)$$

From the relation (3.2) between flux and circulation, the flux of vorticity along the tube is equal to the circulation round any closed curve on the tube wall which encloses the tube once. This quantity is called the strength of the tube and is the most natural measure of its intensity.

Because the strength of the tube does not vary with position along the tube, it follows that vortex tubes are either closed, go to infinity or end on solid boundaries.[6] If the vorticity field is compact, the tubes must be closed or begin and end on boundaries. It is sometimes stated erroneously that vortex lines cannot begin or end in the fluid and therefore either form closed curves or begin and end on boundaries (e.g., Lamb [1932 §145]). In general, a vortex line is infinitely long and passes infinitely often infinitely close to itself even if the field is compact. However, symmetry of the flow field, which is often the case for flows studied analytically, may produce closed vortex lines.[7]

The vorticity field can vanish at an isolated point (a vortex null) without contradicting the statement that vortex lines do not end in the fluid. Examination of the line structure for the case $\omega_i = \alpha_{ij} x_j$, $\alpha_{ii} = 0$, shows that lines either cross or are closed ovals about the isolated point. In the former case, we define the direction at the isolated point as the limit as the point is approached.

In a real fluid satisfying the no-slip boundary condition, vortex lines must be tangential to the surface of a body at rest, except at isolated points of attachment and separation, because the normal component of vorticity vanishes (since the circulation is zero for any circuit on the body). In this case, vortex tubes cannot end on the body and must either be closed or go to infinity at both ends.[8] Vortex tubes can end on the surface of a rotating body. In this case, constraint (v) of §1 is violated; see §2.4 for a discussion

[6] The solenoidal structure of the vorticity field also precludes the tube ending on a singularity in the fluid.
[7] See Truesdell [1954] for a thorough discussion of the geometry of vortex lines.
[8] See Lighthill [1963 Chap. II] for a discussion of the shape of vortex lines in a three-dimensional boundary layer.

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of how the vorticity field can be extended to handle this difficulty (see also Batchelor [1967 §2.4]).

When the vortex tube is immediately surrounded by irrotational fluid, it will be referred to as a vortex filament.[9] Since there is no vorticity outside the filament, it is not necessary to restrict the calculation of the filament strength to curves lying upon its walls; any curve threaded by the filament can be used.

A vortex filament is often just called a vortex, but we shall use this term to denote any finite volume of vorticity immersed in irrotational fluid (with obvious modifications when the flow is two-dimensional). Of course, the vortex filament and vortex require that the fluid is ideal to make strict sense, because viscosity diffuses vorticity, but they are useful approximations for real fluids of small viscosity.

1.5 The laws of vortex motion

Three laws of vortex motion were given by Helmholtz [1858]. For the motion of an ideal barotropic fluid under the action of conservative external body forces,[10] they can be expressed as follows:

I. Fluid particles originally free of vorticity remain free of vorticity.
II. Fluid particles on a vortex line at any instant will be on a vortex line at all subsequent times. Alternatively, it can be said that vortex lines and tubes move with the fluid.
III. The strength of a vortex tube does not vary with time during the motion of the fluid.

The first law implies Lagrange's theorem on the persistence of irrotational motion. Alternative formulations can be given.[11] There are several ways of proving these laws. We shall start with the Euler equations for the conservation of mass and momentum in an ideal fluid moving in the presence of an external force $\mathbf{F}$ per unit mass.

$$\rho_t + \mathbf{u} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{u} = 0, \tag{1}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -(1/\rho) \nabla p + \mathbf{F}. \tag{2}$$

[9] The usage of this expression is neither uniform nor consistent. It is often used to denote an infinitesimal vortex tube. Helmholtz [1858] used it in both senses.

[10] External body forces $\mathbf{F}$ per unit mass are conservative when $\mathbf{F}$ is the gradient of a single-valued scalar, that is, curl $\mathbf{F} = 0$ and $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed, not necessarily reducible, curves in the fluid.

[11] For a historical discussion, see Lamb [1932 §146].

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