Cambridge University Press
0521475724 - Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors
Jacob Palis and Floris Takens
Excerpt
More information

CHAPTER 0

HYPERBOLICITY, STABILITY AND SENSITIVE CHAOTIC DYNAMICAL SYSTEMS

In this chapter we give background information and references to the literature concerning basic notions in dynamical systems that play an important role in our study of homoclinic bifurcations. Essentially, the chapter consists of a summary of the hyperbolic theory of dynamical systems and comments on sensitive chaotic dynamics. This is intended both as an introduction to the following chapters and to provide a more global context for the results to be discussed later and in much more detail than the ones presented in this Chapter 0.

In the first section we concentrate on hyperbolicity and emphasize its intimate relation with various forms of (structural) stability. In the second section we discuss several aspects of sensitivity ("chaos") and indicate how it occurs in hyperbolic systems.

§1 Hyperbolicity and stability

These two concepts, hyperbolicity and (structural) stability, have played an important role in the development of the theory of dynamical systems in the last decades: the hyperbolic theory was mostly developed in the 1960's, having as a main initial motivation the construction of structurally stable systems; in its turn, the notion of structural stability had been introduced much earlier by Andronov and Pontryagin [**AP**,1937]. As conjectured in the late 1960's and only recently proved, it turns out that the two notions are *essentially equivalent* to each other, at least for C^1 diffeomorphisms of a compact manifold. *Of course, for stability one also has to impose the transversality of all stable and unstable manifolds or, for limit-set stability, the no-cycle condition; see concepts below.* It is, however, the hyperbolicity of the limit set which is the main ingredient in this comparison. The solution of this well known conjecture and other related results that we state here go beyond what is needed to understand the next chapters of this book. It is, however, enlightening, in the study of bifurcations (meaning loss of stability), to be acquainted with the fact that the notions of stability and hyperbolicity are that much interconnected.

The concept of (structural) stability deals with the topological persistence of the orbit structure of a dynamical system (endomorphism, diffeomorphism or flow) under small perturbations. (Notice the difference with the concept of Lyapunov stability which concerns attracting sets of a given system). The persistence of the orbit structure is expressed in terms of a *homeomorphism* of the ambient manifold sending orbits of the initial system onto orbits of the perturbed one. If this can be done for any C^k–small ($k \geq 1$) perturbation, then we call the system C^k (*structurally*) *stable* or *globally stable*. Here we are mostly concerned with diffeomorphisms, in which case we require this orbit preserving homeomorphism to be a *conjugacy*. That is, if φ is the initial map, $\tilde{\varphi}$ a C^k–small perturbation of it, we then require the existence of a homeomorphism h such that $h\varphi(x) = \tilde{\varphi}h(x)$ for all x in the ambient manifold. We do not require the conjugacy to be differentiable, for otherwise we would impose invariance of eigenvalues of the linear part of the diffeomorphism at fixed (periodic) points. The same concept applies to endomorphisms, i.e. maps from the ambient manifold to itself.

We will be treating here the case of n-dimensional diffeomorphisms; for diffeomorphisms of the circle (and flows on surfaces) there are early important works of Pliss [**P**,1960], Arnold [**A**,1961a] and specially Peixoto [**P**,1962]. Often, we are concerned with stability restricted to the main part of the orbit structure, the limit set or nonwandering set. Let us recall these concepts.

Let $\varphi: M \to M$ be a C^k ($k \geq 1$) diffeomorphism of a compact, boundaryless, smooth manifold of arbitrary dimension. For $x \in M$, we define the α and ω-limit sets as

$$\alpha(x) = \{y \in M \mid \exists n_i \to -\infty \text{ such that } \varphi^{n_i}(x) \to y\},$$
$$\omega(x) = \{y \in M \mid \exists n_i \to +\infty \text{ such that } \varphi^{n_i}(x) \to y\}.$$

The *positive* and *negative limit sets* are then defined as $L^+(\varphi) = \overline{\bigcup_{x\in M} \omega(x)}$ and $L^-(\varphi) = \overline{\bigcup_{x\in M} \alpha(x)}$; the *limit set* $L(\varphi)$ is the union of $L^+(\varphi)$ and $L^-(\varphi)$. From the definitions, it is clear that $L^+(\varphi)$ and $L^-(\varphi)$ are φ-invariant, i.e. $\varphi(L^+(\varphi)) = L^+(\varphi)$ and $\varphi(L^-(\varphi)) = L^-(\varphi)$. Moreover, for each $x \in M$, $\varphi^n(x)$ approaches $L^+(\varphi)$ or $L^-(\varphi)$ as $n \to \infty$ or $n \to -\infty$. So $L^+(\varphi)$ and $\varphi|L^+(\varphi)$, $L^-(\varphi)$ and $\varphi|L^-(\varphi)$, describe the asymptotic behaviour of orbits, i.e. sequences $\{\varphi^n(x)\}$, in M for $n \to \infty$ or $n \to -\infty$. Another relevant concept is that of *nonwandering* point: x is nonwandering if for any neighbourhood U of it, there is an integer n such that $\varphi^n(U) \cap U \neq \phi$. Again, the union of the nonwandering points, which is denoted by $\Omega(\varphi)$, is a φ-invariant compact set. Clearly, all α or ω-limit points as well as homoclinic points (see Chapter 1) are nonwandering. In Section 4, Chapter 5, we provide an example of a homoclinic tangency which is in $L^+(\varphi)$ but not in $L^-(\varphi)$, or vice versa; this example also shows that, in general, the nonwandering and limit sets are different. However, as we shall see in Chapter 2, any transversal homoclinic point is an accumulation point of periodic orbits and so it is in $L^+(\varphi)$, in $L^-(\varphi)$, and in $\Omega(\varphi)$. Finally, we define another useful concept: the chain recurrent set, which is the union of the chain recurrent points. A point p is chain recurrent if for each $\varepsilon > 0$ there are points $x_0 = p, x_1, x_2, \cdots, x_k = p$ such that $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leq i \leq k$, d being a distance function.

If $L^+(\varphi)$ (or $L^-(\varphi)$) is hyperbolic (see Chapter 2 and Appendix 1), then one can show that $\overline{\text{Per}\,(\varphi)} = L^+(\varphi)$ (or $L^-(\varphi)$), where $\text{Per}\,(\varphi)$ indicates the set of periodic points of φ; one can then write as in [**N**,1972]:

$$L^+(\varphi) = \Lambda_1 \cup \cdots \cup \Lambda_k$$

where each Λ_i is *invariant*, *compact*, *transitive* (it has a dense orbit) and has a *dense subset of periodic orbits*. This is called the spectral decomposition of $L^+(\varphi)$. Moreover, by [**HPPS**, 1970] (see also [**N**,1980], [**B**,1977] for a different and relevant proof using the idea of "shadowing" of orbits), each Λ_i is the *maximal invariant set* in a neighbourhood of it. This last fact is actually equivalent to what we call *local product structure* in Λ_i: there exist $\varepsilon > 0$ and $\delta > 0$ such that if the distance between $x, y \in \Lambda_i$ is smaller than δ then their local stable and unstable manifolds of size ε (see Appendix 1) intersect each other in a unique point and this point is in Λ_i. Also, one can prove that if $\omega(x) \subset \Lambda_i$ then $x \in W^s(z)$ for some $z \in \Lambda_i$. In general, a set with the properties above is called a *basic set* for the diffeomorphism.

If we assume that the nonwandering set $\Omega(\varphi)$ is hyperbolic and $\overline{\text{Per}\,(\varphi)} = \Omega(\varphi)$, then we say that φ satisfies *Axiom A*. In this case we have $\Omega(\varphi) = L^+(\varphi)$ and so we can write the nonwandering set as a finite union of basic sets. This is the content of Smale's spectral decomposition theorem [**S**,1970]; the corresponding version for the limit set as presented above appeared later in [**N**,1972]. Notice that if $\Lambda_1 \cup \cdots \cup \Lambda_k$ is the spectral decomposition of $L^+(\varphi)$ (or $\Omega(\varphi)$) then $M = \bigcup_i W^s(\Lambda_i)$, where $W^s(\Lambda_i) = \{y \mid \omega(y) \subset \Lambda_i\}$ is called the stable set of Λ_i; as discussed above $W^s(\Lambda_i) = \bigcup_{x \in \Lambda_i} W^s(x)$. Similar statements are valid for the unstable sets of the Λ_i's, corresponding to a spectral decomposition of $L^-(\varphi)$ or $\Omega(\varphi)$. Some $W^s(\Lambda_i)$ must be open; in this case Λ_i is called an *attractor*. (A more general definition of attractor is in the next section). Dually if $W^u(\Lambda_i)$ is open, then we say that Λ_i is a *repeller*. Finally, Λ_i is of *saddle type* if it is neither an attractor nor a repeller. Another property of Axiom A diffeomorphisms: the stable sets of attractors cover an open and dense subset of M and the same is true for unstable sets of repellers. It is an interesting fact that if φ is C^2, then the union of the stable sets of attractors has total Lebesgue measure; see Ruelle [**R**,1976] and Bowen–Ruelle [**BR**,1975]. There are, however, examples of C^1 saddle-type basic sets with stable sets of positive Lebesgue measure [**B**,1975b], which are detailed in Chapter 4. Another interesting fact about basic sets is that they are *expansive*: for each basic set Λ there is a constant $\alpha > 0$ such that for each pair of different points in Λ, their (full) orbits get apart by at least α. *From this it follows that hyperbolic attractors which are not just fixed or periodic sinks have sensitive dependence on initial conditions*: for most pairs of different points in the stable set of such an attractor Λ, the positive orbits get apart by at least a constant (which depends on the attractor). *Most* here means probability 1 in $W^s(\Lambda) \times W^s(\Lambda)$.

The following relevant result concerning basic sets states that they are persistent under C^k−small perturbations (see Appendix 1); in particular, hyperbolic attractors are persistent.

THEOREM 1. *If Λ is a basic set for a C^k diffeomorphism $\varphi\colon M \to M$, then for any $\tilde{\varphi}$ close to φ its maximal invariant set in some neighbourhood of Λ is a basic set $\tilde{\Lambda}$ and $\varphi|\Lambda$ is conjugate to $\tilde{\varphi}|\tilde{\Lambda}$. Moreover, if we require the conjugacy to be C^0−close to the inclusion map of Λ into M, then it is unique and it is in fact Hölder continuous.*

Usually we call this set $\tilde{\Lambda}$ *the "smooth" or "analytic" continuation of* Λ for a given perturbation of φ. The result can be applied when all of M is hyperbolic for φ; in this case we say that φ is Anosov. As a corollary, we have the following

THEOREM 2 [**A**,1967]. *Anosov diffeomorphisms are globally C^k-stable.*

Moser's elegant proof of this last result [**M**,1969] actually suggested the original proof of the first theorem, but of course Anosov's theorem was shown before. Nowadays there is a simple way of showing the existence of the conjugacy: one uses again the idea of shadowing of orbits mentioned above.

Another relevant class of systems in our context is that of Morse–Smale diffeomorphisms. We call φ Morse-Smale if

(i) $\Omega(\varphi)$ consists of a finite number of periodic orbits, all of them hyperbolic,

(ii) φ satisfies the transversality condition: the stable and unstable manifolds of the elements in $\Omega(\varphi)$ are all in general position.

It turns out that in (i) above one can write $L^+(\varphi)$ or $L^-(\varphi)$ instead of $\Omega(\varphi)$. *Morse–Smale diffeomorphisms are abundant in the sense that they contain the time-1 maps of an open and dense subset of gradient vector fields on every manifold.*

THEOREM 3 [**PS**,1970]. *Morse–Smale diffeomorphisms are C^k-stable. (In particular, there are stable diffeomorphisms on every manifold.) Conversely,* *among diffeomorphisms whose nonwandering sets consist of finitely many periodic orbits, hyperbolicity of these orbits and transversality of their stable and unstable manifolds are necessary for C^k-stability.*

In view of these results, it seemed that hyperbolicity of the nonwandering set (or limit set) and transversality of stable and unstable manifolds were the precise conditions that should grant C^k-stability of the diffeomorphism. So, let us say that an Axiom A diffeomorphism f satisfies the transversality condition if the stable and unstable manifolds of any two points in $\Omega(f)$ are in general position.

STABILITY CONJECTURE [**PS**,1970]: A C^k (or $C^s, s \geq k$) diffeomorphism is C^k-stable if and only if it satisfies Axiom A and the transversality condition.

REMARK 1: We can phrase the stability conjecture in terms of the limit set (or just the positive or negative limit set): a diffeomorphism is C^k-stable if and only if its limit set is hyperbolic and the transversality condition holds. This equivalent, and perhaps more elegant, statement is further commented on below.

Much in parallel, let us start discussing stability restricted to the nonwandering set or to the limit set. We say that a diffeomorphism φ is $C^k-\Omega$-stable if there exists a C^k neighbourhood of it such that if $\tilde{\varphi}$ belongs to this neighbourhood, then $\varphi|\Omega(\varphi)$ is conjugate to $\tilde{\varphi}|\Omega(\tilde{\varphi})$. Similarly for the limit set. Let now φ be an Axiom A diffeomorphism and let $\Omega(\varphi) = \Lambda_1 \cup \cdots \cup \Lambda_k$ be its spectral decomposition. A j-cycle on Ω is a string of j pairs of points $x_1, y_1 \in \Lambda_{i_1}, \cdots, x_j, y_j \in \Lambda_{i_j}$, with not all $i_1, \cdots, i_j$ equal, such that $W^u(y_1) \cap W^s(x_2) \neq \phi, \cdots, W^u(y_j) \cap W^s(x_1) \neq \phi$. If the limit set, or just the positive or negative limit set, is hyperbolic then we also have a spectral decomposition for it and the notion of cycles can be applied. In either case, when there are *no cycles* we can construct a *filtration* ([**S**,1967],[**C**,1978]): a sequence $M_0 = \phi$, $M_1 \subset M_2 \subset \cdots \subset M_k = M$ of compact submanifolds with boundary for $0 < i < k$ such that $\varphi(M_i) \subset$ Int M_i and in $M_{i+1} - M_i$ the maximal invariant set is Λ_i. The existence of filtrations implies the following proposition, which clarifies why in the statement of stability results and conjectures we can mention either Axiom A (hyperbolicity of the nonwandering set and density of periodic orbits) or just hyperbolicity of the limit set or the chain recurrent set; see [**N**,1972; **FS**,1977].

PROPOSITION 1. *If $L(\varphi)$ (or just $L^+(\varphi)$ or $L^-(\varphi)$) is hyperbolic and there are no cycles on it, then $\Omega(\varphi) = L(\varphi)$ and so φ satisfies Axiom A (and there are no cycles on $\Omega(\varphi)$). If the chain recurrent set of φ is hyperbolic then φ satisfies Axiom A and the no-cycle condition.*

The existence of a filtration implies a global control on the nonwandering or limit set when we perturb the map: there appear no nonwandering points far from the original ones. This fact and the persistence of basic sets stated above imply Smale's Ω-stability theorem [**S**,1970].

THEOREM 4. *The set of diffeomorphisms satisfying Axiom A and the no-cycle property is open in Diff$^k(M)$ and its elements are Ω-stable.*

In the way of a converse to this theorem, we have the following

THEOREM 5 [**P**,1970]. *If φ is an Axiom A diffeomorphism and there is a cycle on $\Omega(\varphi)$, then φ is not Ω-stable. In fact, there are arbitrarily close diffeomorphisms $\tilde{\varphi}$ such that Per $(\varphi) \not\subseteq$ Per $(\tilde{\varphi})$. A similar statement is true for diffeomorphism with hyperbolic limit sets.*

COROLLARY 1. *If φ and all nearby diffeomorphisms have their nonwandering sets (limit sets) hyperbolic, then they are Ω-stable. On the other hand, if φ or arbitrarily close diffeomorphisms exhibit homoclinic tangencies (see Chapter 2 or 3) then φ is not Ω-stable.*

In view of the results above we formulate the following conjecture.

Ω-STABILITY CONJECTURE: A diffeomorphism φ is $C^k - \Omega$-stable if and only if it satisfies Axiom A and the no-cycle property.

Back to the stability conjecture, in the early 1970's Robbin [**R**,1971] proved that diffeomorphisms satisfying Axiom A and the transversality condition are C^k-stable for $k \geq 2$. Soon afterwards, de Melo [**M**,1973b] proved the result for C^1 diffeomorphisms of surfaces using ideas close to [**PS**, 1970]. By 1976, Robinson [**R**,1976] completed the result for $k = 1$ in any dimension using an approach somewhat different from the previous ones. Before, in [**R**,1974], he had the corresponding version for flows with $k \geq 2$. In another paper [**R**,1973], he also proved that the transversality condition is necessary for C^k stability.

So, in both the stability and the Ω-stability conjectures it remained to show that hyperbolicity of the nonwandering set was necessary for either kind of stability. This was the missing fact to establish such a fundamental link between stability or Ω-stability and hyperbolicity of the nonwandering (or limit) set. From the beginning it was clear that with the available knowledge this goal was probably beyond reach for C^k stability or Ω-stability when $k \geq 2$. In fact, it is still unknown whether $\overline{\text{Per}\ (\varphi)} = \Omega(\varphi)$ if φ is $C^2 - \Omega$-stable; for $k = 1$, this follows from Pugh's closing lemma [**P**,1967]. By 1980, Mañé concluded both conjectures for C^1 surface diffeomorphisms [**M**,1982]; independently, Liao [**L**,1980] and Sannami [**S**,1983] obtained this same result. For flows, Liao seems to have made substantial progress toward the same conclusion on 3-manifolds; in higher dimensions the question is still open. It is interesting to observe that the situation looks rather different for manifolds with boundary and flows leaving the boundary invariant: there exist singular horseshoes that are stable but not hyperbolic [**LP**,1986].

Finally, just about 20 years after it was proposed, Mañé in a remarkable paper [**M**,1988] presented a solution of the C^1 stability conjecture for diffeomorphisms in any dimension. His proof, however, did not include the $C^1 - \Omega$-stability conjecture because he needed the transversality condition (typical of stability but not of Ω-stability) to complete his arguments. This was done in [**P**,1988], arguing instead just with the no-cycle property.

Actually, one may ask if the following set of equivalences is true:

$f \in \mathcal{D}^1(M) \iff f$ satisfies Axiom A and the no-cycle property $\iff f$ is Ω-stable.

Here, $\mathcal{D}^1(M)$ denotes the interior, with respect to the C^1 topology, of the set of diffeomorphisms whose periodic orbits are all hyperbolic. From the results above it only remained to show that if f is in $\mathcal{D}^1(M)$ then it satisfies Axiom A. The truth of this statement, and thus of the set of equivalences above, was recently and independently announced by Aoki [**A**,1991] and Hayashi [**H**,1991].

Closing this section, we want to pose two questions that are somewhat inspired by the results above and are relevant in the context of homoclinic bifurcations, the main topic of this text. They concern differentiable arcs or one-parameter families φ_μ of C^k diffeomorphisms such that φ_μ satisfies Axiom A and the transversality condition for $\mu < \mu_0$ and φ_{μ_0} is Ω-unstable; such μ_0 is called the *first bifurcation point* of φ_μ.

The first question is: what types of bifurcation can occur for φ_{μ_0} if the family φ_μ is *generic*, i.e. if the family φ_μ belongs to some *residual (Baire second category)* class of families ? We *conjecture* that in *two dimensions*, we only have three possibilities: φ_{μ_0} has a nonhyperbolic periodic orbit, a homoclinic tangency, or a heteroclinic tangency involving periodic points in a cycle. *In higher dimensions* there are more cases, like *homoclinic tangencies of basic sets:* $W^s(x)$ tangent to $W^u(y)$ for x, y in the same basic set, neither point being necessarily periodic. A main difference is that the *boundary* of a basic set in two dimensions is made of stable and unstable manifolds of periodic orbits; see Appendix 2.

The next question concerns a generic family φ_μ on a surface such that φ_μ is Morse–Smale for $\mu < \mu_0$ and there exist values $\tilde{\mu} > \mu_0$ arbitrarily near μ_0 such that $\varphi_{\tilde{\mu}}$ has infinitely many periodic orbits. The problem now is whether there is $\mu_1 > \mu_0$ near μ_0 so that φ_{μ_1} exhibits some homoclinic tangency associated to some periodic point (see Chapter 3). In higher dimensions we formulate the question in terms of homoclinic bifurcations as in Chapter 7: one may persistently (open set of families) create homoclinic orbits without creating homoclinic tangencies at all!

We refer the reader to [**NP**,1976; **NP**,1973] where similar questions were studied and a version of the above conjecture posed.

§2 Sensitive chaotic dynamics

The notion of "*chaos*" in dynamical systems, as opposed to theology and the usual meaning of total disorder, refers to a situation where (forward) orbits do not converge to a periodic or quasi-periodic orbit and where *the evolution of the orbits has some degree of unpredictability* or *their behaviour is sensitive with respect to initial conditions*. Although this phenomenon was theoretically known, in particular for nontrivial hyperbolic attractors, it came to many as a surprise that it also appeared in numerically generated orbits of quite simple systems. Among the first such examples, which were investigated numerically, there were the Lorenz attractor [**L**,1963], the logistic map [**CE**,1980] and the Hénon map [**H**,1976]–in fact all these systems depend on parameters and for a substantial set of parameter values these sensitive or chaotic phenomena appear. All these examples are nonhyperbolic and a big effort was needed to get some theoretical understanding of them. The last two examples play an important role in the dynamics at homoclinic bifurcations; see Chapters 3, 6 and 7. In this section we shall formalize the main notions involved in "chaos" and indicate instances where they occur in hyperbolic systems. We want to point out that there are different formalizations of these notions (although there is agreement about the general flavour).

Here, we restrict ourselves to dynamical systems defined by a continuous map $\varphi: M \to M$, where we assume M to be a compact metric space—if M is not compact the discussion still makes sense if we restrict to points in M whose positive orbits have a compact closure and, hence, are bounded. We say that the (positive) *orbit* $\{x, \varphi(x), \varphi^2(x), \cdots\}$ of x is *sensitive* or *chaotic* if there is a positive constant $C > 0$ such that for any $q \in \omega(x)$ (for the definition see the previous section) and any $\varepsilon > 0$ there are integers $n_1, n_2, n > 0$ such that $d(\varphi^{n_1}(x), q) < \varepsilon$, $d(\varphi^{n_2}(x), q) < \varepsilon$, but $d(\varphi^{n_1+n}(x), \varphi^{n_2+n}(x)) > C$. We observe that an orbit asymptotic to a (quasi-) periodic one is not chaotic in the above sense: for such orbits, if $\varphi^{n_1}(x)$ and $\varphi^{n_2}(x)$ are close, then $\varphi^{n_1+n}(x)$ and $\varphi^{n_2+n}(x)$ remain close for all $n > 0$. Also, a sensitive orbit in the above sense is unpredictable to the extent that if we know that some y on the positive orbit of x is extremely close to $q \in \omega(x)$, this is not enough to predict, say within distance C, all future iterates of y. *This last fact is closely related with the sensitive dependence on initial conditions discussed in the previous section.* Similarly to the fact that in the stable set of a *nontrivial hyperbolic attractor* one has sensitive dependence on initial conditions, one can prove that most points in such a stable set have *sensitive orbits*–the set of points with chaotic orbits even has *total Lebesgue measure* in the stable set; see [**ER**, 1985]. Not only in hyperbolic attractors are there sensitive orbits, but also in nontrivial hyperbolic sets of saddle-type: this can easily be seen using symbolic dynamics as introduced for the horseshoe example in Chapter 2. The difference is that in this last case the set of points with sensitive orbits has Lebesgue measure zero.

For the case where M is a manifold, and hence the notion "Lebesgue measure zero" is defined, we say that the *dynamical system* defined by φ is *sensitive*, or that φ has *sensitive* or *chaotic dynamics*, if the set of points with sensitive or chaotic orbits has *positive Lebesgue measure*.

Up to now, the notion of sensitive or chaotic dynamics has mainly been defined negatively (?!): namely, in terms of unpredictability. From the numerical experiments, mentioned before, it was however apparent that certain aspects of the positive orbits were very predictable: for any initial point (in some open set), one always finds numerically the *same* ω-limit set. Thus, chaos or sensitivity as observed above should not be interpreted as *total unpredictability*: instead, it should mean that most orbits of the system will finally be getting apart from each other and continue to wander around a "larger" but definite set. This leads to the definition of a *strange attractor*. We say that a compact set $A \subset M$ is a *strange attractor* if there is an open set U with a subset $N \subset U$ of Lebesgue measure zero, such that for all $x \in U \setminus N$, $\omega(x) = A$ and the (positive) orbit of x is chaotic. (We allow for the exceptional set N because even for hyperbolic attractors a dense set of points, of Lebesgue measure zero, is attracted to periodic orbits; and *as long as N has Lebesgue measure zero it should not be of influence on numerical experiments*). Another and even more common definition of *strange attractor* is to require A to have sensitive dependence on initial conditions on U; see the discussions in Chapter 7.

In many cases, including the numerical examples, one is interested in the persistence of phenomena not only under a (small) perturbation of the initial point (of a positive orbit) but also under a small perturbation of the map $\varphi\colon M \to M$. Intuitively, one says that the dynamics of φ is *persistently sensitive or chaotic* if small perturbations of φ have, with positive probability, sensitive dynamics. But the problem with this intuitive notion is that there is no "natural" measure on the set of maps $\varphi\colon M \to M$. On the other hand, if we would require all small perturbations of φ to have sensitive dynamics (which is the case for a nontrivial hyperbolic attractor and whose dynamics is called for this reason *fully persistently sensitive or chaotic*) we would exclude important cases like the *logistic map* for many values of the parameter. There is, however, one important instance in which this notion can be formally defined: if we are in a context where the notion of generic k-parameter unfoldings $\varphi_{\mu_1,\dots,\mu_k}$ of φ is defined (with $\varphi_{0,\dots,0} = \varphi$), we say that φ has persistently sensitive dynamics if for any such generic k-parameter unfolding, the set of $\mu = (\mu_1, \dots, \mu_k)$ values, for which φ_μ has sensitive dynamics, has positive Lebesgue measure.

In the last section of Chapter 6 and in Chapter 7 we shall discuss the consequences of our investigations of homoclinic bifurcations in terms of the above notions of sensitive or chaotic dynamics.