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978-0-521-47423-8 - Automorphic Representations and L-Functions for the General Linear Group, Volume I

Dorian Goldfeld and Joseph Hundley

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