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AUTOMORPHIC REPRESENTATIONS AND L-FUNCTIONS FOR THE GENERAL LINEAR GROUP

Volume I

This graduate-level textbook provides an elementary exposition of the theory of automorphic representations and L-functions for the general linear group in an adelic setting. Definitions are kept to a minimum and repeated when reintroduced so that the book is accessible from any entry point, and with no prior knowledge of representation theory. The book includes concrete examples of global and local representations of $GL(n)$, and presents their associated L-functions.

In Volume I, the theory is developed from first principles for $GL(1)$, then carefully extended to $GL(2)$ with complete detailed proofs of key theorems. Several proofs are presented for the first time, including Jacquet's simple and elegant proof of the tensor product theorem. In Volume II the higher rank situation of $GL(n)$ is given a detailed treatment.

Containing over 250 exercises written by Xander Faber, this book will motivate students to begin working in this fertile field of research.

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Automorphic Representations and L-Functions for the General Linear Group

Volume I

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CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521474238

First published 2011

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

ISBN 978-0-521-47423-8 Hardback

To Ada, Dahlia, and Iris.
–D. G.

To Melissa, and to my Family.
–J. H.

Cambridge University Press
978-0-521-47423-8 - Automorphic Representations and L-Functions for the General Linear Group, Volume I
Dorian Goldfeld and Joseph Hundley
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Introduction

The theory of L-functions is an old subject with a long history. In the 1940s Hecke and Maass rewrote the classical theory in the setting of automorphic forms, and it seemed as if the theory of L-functions had settled into a fairly final form. This view was effectively overturned with the publication of two major books: [Gelfand-Graev-Piatetski-Shapiro, 1969], [Jacquet-Langlands, 1970], where it was shown that the theory of L-functions could be recast in the language of infinite dimensional complex representations of reductive groups.

Another milestone in the recent theory of L-functions was the book by Roger Godement and Hervé Jacquet, [Godement-Jacquet, 1972], which defined for the first time the standard L-functions attached to automorphic representations of the general linear group, and proved their key properties by generalizing the seminal ideas of [Tate, 1950], [Iwasawa, 1952, 1992]. The proofs in [Godement-Jacquet, 1972] made fundamental use of matrix coefficients associated to automorphic representations. The standard L-functions of the general linear group are often called Godement-Jacquet L-functions. Although several other techniques have since been discovered to obtain the main analytic properties of such L-functions, none is more beautiful and elegant than the method of matrix coefficients, originally devised by Godement and Jacquet, which is a major theme of this book.

Modern research in the theory of automorphic representations and L-functions is largely focused in the direction of the Langlands program. Quoting from [Bernstein-Gelbart, 2003]:

> *The Langlands program roughly states that, among other things, any L-function defined number-theoretically is the same as the one which can be defined as the automorphic L-function of some* $GL(n)$*. In this loose way, every L-function is (conjecturally) viewed as one and the same object.*

Langlands' philosophy established the central importance of the general linear group for number theory. A great step forward was obtained recently when Ngo proved the fundamental lemma (see: [Laumon-Ngo, 2004], [Ngo, 2008]).

The purpose of this book is to provide an elementary yet extremely rigorous exposition of the theory of cuspidal automorphic representations and L-functions for the general linear group in a textbook form that can be understood by the beginning graduate student with minimal background in representation theory. The theory of Eisenstein series and the $\mathcal{L}^2$ decomposition of the space of automorphic forms are omitted for reasons of space. To simplify the presentation, the theory is restricted to the adele group of $\mathbb{Q}$, although in most cases, the proofs can be easily generalized to any number field. Definitions are reintroduced where necessary so that the book is easily accessible from any entry point. Most definitions and key ideas are explained in concert with simple concrete examples. Almost every definition, theorem, and proposition is captioned, so that the flow of ideas is easy to grasp. The book contains over 250 exercises as well as over 50 pages of solutions to exercises.

The first chapter introduces the theory of p-adic fields and the adele ring $\mathbb{A}_\mathbb{Q}$ from first principles. A highlight of this chapter is a short, rigorous, and elementary proof of the Fourier expansion of periodic adelic functions which plays such a crucial role in the proof of the analytic continuation and functional equation of L-functions.

The second chapter presents the theory of automorphic representations and L-functions for the group $GL(1, \mathbb{A}_\mathbb{Q})$. This is essentially Tate's thesis recast in the language of automorphic representations for $GL(1)$.

Chapters 3 through 11 develop the theory of automorphic representations and L-functions for $GL(2)$. Particular care is taken to show the relationship between irreducible cuspidal automorphic representations of $GL(2, \mathbb{A}_\mathbb{Q})$ and Hecke-Maass newforms for congruence subgroups of $SL(2, \mathbb{Z})$. Highlights include the classification of the irreducible admissible representations of $GL(2, \mathbb{Q}_p)$ given in Chapter 6, the classification of irreducible admissible $(\mathfrak{g}, K_\infty)$-modules given in Chapter 7, growth of matrix coefficients given in Chapter 8, Jacquet's simple and extremely elegant proof of the tensor product theorem in Chapter 10, and the proofs (using matrix coefficients) of the key analytic properties of the Godement-Jacquet L-functions given in Chapter 11.

Finally, the entire theory is redone for the more general case of $GL(n)$ in the final Chapters 12 through 15. Chapter 12 presents a classical theory of automorphic forms for $GL(n, \mathbb{A}_\mathbb{Q})$, which generalizes the theory presented in [Goldfeld, 2006]. Instead of K-fixed forms, automorphic forms with arbitrary K-type, level, and character are studied. Chapter 14 presents the Bernstein-Zelevinsky classification of the smooth irreducible representations of $GL(n, \mathbb{Q}_p)$ as well as Vogan's classification of the irreducible unitary representations of $GL(n, \mathbb{R})$. The book ends with the theory of the Godement-Jacquet L-function for $GL(n)$.

By seeing the theory of automorphic representations and L-functions in three different settings:

- *the abelian setting of* $GL(1)$*;*
- *the rank one setting of* $GL(2)$*;*
- *the higher rank situation of* $GL(n)$;

along with many simple concrete examples to investigate, the beginning student can gain deep insight into this beautiful subject. It is hoped that by reading this book students and researchers will be motivated to begin working in this fertile field of research.

The authors are deeply indebted to Hervé Jacquet for walking them through the most difficult steps and showing them new proofs of many results. These proofs substantially simplify the arguments previously available in the literature, and we would like to thank Hervé Jacquet for allowing us to include them in this exposition. Without his help this book could not have been written. We are especially grateful to Xander Faber for a careful reading of the manuscript, pointing out innumerable errors, and for creating all the exercises and a solutions section, which will be so invaluable for students. The authors would like to specially thank Min Lee for carefully reading the book, correcting many errors, and preparing the index and table of symbols. We would like to thank Gautam Chinta, Ivan Fesenko, Joe Pleso, and Shou-Wu Zhang for many helpful comments. We thank Jacqueline Anderson, Atanas Atanasov, Alberto Baider, Ioan Filip, Timothy Heath, Jeffrey Hoffstein, Thomas Hulse, Eren Mehmet Kiral, Karol Koziol, Chan Ieong Kuan, Li Mei Lim, Matthew Spencer, and Ian Whitehead for patiently reading various chapters and pointing out errors and typos. We thank the NSF and NSA for financial support. Finally we want to thank Roger Astley and Cambridge University Press for encouraging us to write and publish this book.

Dorian Goldfeld and Joseph Hundley

Preface to the Exercises

My goal for this project was to remedy some of my ignorance of the theory of automorphic forms. I hope these exercises will aid the reader in doing the same. If an exercise requires some sort of inspiration that isn't immediately obvious from the text, then I have tried to give at least a hint. I have attempted to write a detailed sketch or a full solution whenever an exercise was particularly difficult (for me). But it will be evident that I have violated both of these guiding principles at times with little rhyme or reason. An exercise marked with a * is particularly tricky (again, for me).

My thanks go to Dorian for the opportunity to be a part of this project, and to Joe for patiently answering loads of my questions. It's been a pleasure working with both of you. A National Science Foundation Postdoctoral Research Fellowship provided my funding during the completion of this project. Finally, I would like to thank my wife, Alana, for her unwavering support of my endeavors, especially those that detract from our time together.

— Xander Faber