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Adeles over $\mathbb{Q}$

1.1 Absolute values

Definition 1.1.1 (Absolute value) An absolute value on a field F is a non-negative real valued function $|\ |$ on F which satisfies the conditions:

(i) $|x| = 0$ if and only if $x = 0$,
(ii) $|xy| = |x| \cdot |y|$,
(iii) $|x + y| \leq |x| + |y|$, (triangle inequality)

for all $x, y \in F$.

If an absolute value $|\ |$ on a field F satisfies the stronger condition

$$|x + y| \leq \max\big(|x|, |y|\big), \tag{1.1.2}$$

then it is called a non-archimedean absolute value. If condition (1.1.2) fails for some $x, y \in F$, then $|\ |$ is called an archimedean absolute value.

It is always possible to define a trivial absolute value $|\ |_{\text{trivial}}$ on any field F where

$$|x|_{\text{trivial}} = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $|\ |_{\text{trivial}}$ is not very interesting, we shall usually exclude it in our discussions.

Definition 1.1.3 (Equivalence of absolute values) Two absolute values $|\ |_1$ and $|\ |_2$, defined on the same field F, are termed equivalent if there exists $c > 0$ such that $|x|_1 = |x|_2^c$ for all $x \in F$.

Example 1.1.4 The field $\mathbb{Q}$ of rational numbers has the classical (and very ancient) archimedean absolute value which we denote by $|\ |_\infty$ which is defined by

$$|x|_\infty = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0, \end{cases} \tag{1.1.5}$$

for all $x \in \mathbb{Q}$. For each prime p one may define the non-archimedean absolute value $|\ |_p$ as follows. Given $x \in \mathbb{Q}$ with $x = p^k \cdot \frac{m}{n}$ with $p \nmid mn$, and $k \in \mathbb{Z}$, we define

$$|x|_p = \left| p^k \cdot \frac{m}{n} \right| = p^{-k}. \tag{1.1.6}$$

The definition of $|\ |_p$ has the effect that the non-archimedean absolute values of numbers divisible by high powers of p become small.

Theorem 1.1.7 (Ostrowski) *The only non-trivial absolute values on $\mathbb{Q}$ are those equivalent to the $|\ |_p$ or the ordinary absolute value $|\ |_\infty$.*

Proof See [Cassels, 1986], [Murty, 2002]. □

Theorem 1.1.8 (Product formula) *Let $\alpha \in \mathbb{Q}$ with $\alpha \neq 0$. The absolute values $|\ |_v$, given by (1.1.5), (1.1.6), satisfy the product formula*

$$\prod_v |\alpha|_v = 1$$

where the product is taken over all $v \in \{\infty, 2, 3, 5, 7, 11, 13, \dots\}$, i.e., $v = \infty$ or v is a prime.

Proof The proof is elementary and left to the reader. □

Definition 1.1.9 (Finite and infinite primes) Following the modern tradition we shall call $v = 2, 3, 5, 7, 11, 13 \dots$ the finite primes and $v = \infty$ the "infinite "or" archimedean prime." Henceforth, we shall adhere to the convention that v refers to an arbitrary prime v (with v finite or infinite), while p refers specifically to a finite prime.

1.2 The field $\mathbb{Q}_p$ of p-adic numbers

An absolute value $|\ |$ on a field F allows us to define the notion of distance between two elements $x, y \in F$ as $|x - y|$. We may also introduce a topology on F where the basis of open sets consists of the open balls $B_r(a)$ with center $a \in F$ and radius $r > 0$:

$$B_r(a) = \{x \mid |x - a| < r\}.$$

A sequence of elements $x_1, x_2, x_3, \dots \in F$ is termed Cauchy provided

$$|x_m - x_n| \longrightarrow 0, \qquad (m, n \to \infty). \tag{1.2.1}$$

A field F with a non-trivial absolute value $|\ |$ is said to be complete if all Cauchy sequences of elements $x_1, x_2, x_3, \dots \in F$ have the property that there

exists an element $x^* \in F$ such that $|x_n - x^*| \to 0$ as $n \to \infty$, i.e., all Cauchy sequences converge.

If a field F is not complete, it is possible to complete it by standard methods of analysis. In brief, one adjoins to the incomplete field F all the elements arising from equivalence classes of Cauchy sequences, where two Cauchy sequences $\{x_1, x_2, \dots\}$, $\{y_1, y_2, \dots\}$ are equivalent if $\lim_{i\to\infty} |x_i - y_i| = 0$. The original elements $\alpha \in F$ are then realized as the equivalence class of the constant Cauchy sequence $\{\alpha, \alpha, \alpha, \dots\}$. Addition, subtraction, and multiplication of the representatives $\{x_i\} = \{x_1, x_2, \dots\}$, $\{y_i\} = \{y_1, y_2, \dots\}$ of two equivalence classes of Cauchy sequences are defined by

$$\{x_i\} \pm \{y_i\} = \{x_i \pm y_i\}, \qquad \{x_i\} \cdot \{y_i\} = \{x_i \cdot y_i\}.$$

The definition of division is the same, except one has to be careful to not divide by zero because in a Cauchy sequence $\{x_1, x_2, x_3, \dots\}$, some of the x_i may be 0. Happily, this is not a problem, because every Cauchy sequence is equivalent to a Cauchy sequence without any zero terms and we always choose such a representative for performing division. The sequence of quotients will be Cauchy, provided the Cauchy sequence by which we divide does not converge to zero.

Definition 1.2.2 (*p*-adic fields) Let p be a prime number. The completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\ |_p$, defined by (1.1.6), is denoted as $\mathbb{Q}_p$ and called the p-adic field.

We now present two explicit constructions of $\mathbb{Q}_p$.

Analytic construction of $\mathbb{Q}_p$: The first construction we present is based on the notion of Cauchy sequences. Let $k < n$ be any two integers (positive or negative) and for each i satisfying $k \le i \le n$ let $0 \le a_i < p$ also be an integer. If we assume $a_k \neq 0$, then it easily follows from (1.1.6) that

$$\left| \sum_{i=k}^{n} a_i p^i \right|_p = p^{-k}. \tag{1.2.3}$$

Fix $k \in \mathbb{Z}$. An infinite sequence $\{a_k, a_{k+1}, a_{k+2}, \dots\}$, where $a_i \in \{0, 1, \dots, p-1\}$ for each $i \ge k$, and $a_k \neq 0$, determines an infinite sequence

$$\begin{aligned} x_1 &= a_k p^k \\ x_2 &= a_k p^k + a_{k+1} p^{k+1} \\ x_3 &= a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} \\ &\vdots \end{aligned}$$

of elements in $\mathbb{Q}$. By (1.2.3) it is easy to see that the sequence $x_1, x_2, x_3 \ldots$ is a Cauchy sequence. Formally, we may define

$$\lim_{i\to\infty} x_i = \sum_{i=k}^{\infty} a_i p^i, \qquad (\text{with } |x_i|_p = p^{-k} \text{ for all } i = 1, 2, \ldots).$$

Let $\mathbb{Z}_p$ denote the set of all elements x of the completed field $\mathbb{Q}_p$ which satisfy $|x|_p \leq 1$. By (1.1.2) it easily follows that $\mathbb{Z}_p$ must be a ring with maximal ideal

$$\pi = \left\{ x \in \mathbb{Z}_p \;\middle|\; |x|_p < 1 \right\}.$$

It is easy to check that $\pi = p \cdot \mathbb{Z}_p$. Every $x \in \mathbb{Z}_p$ can be uniquely realized as the equivalence class of a Cauchy sequence of the form

$$\left\{ a_0, \quad a_0 + a_1 p, \quad a_0 + a_1 p + a_2 p^2, \quad a_0 + a_1 p + a_2 p^2 + a_3 p^3, \quad \ldots \right\}$$

where $0 \leq a_i < p$ for $i = 0, 1, 2, \ldots$. One may check this by first showing that every element of $\mathbb{Z}_p$ contains a sequence consisting entirely of integers. Every integer may be expressed as a finite sum $a_0 + \cdots + a_N p^N$. One then shows that for the sequence to be Cauchy, the "digit" a_i must be eventually constant for each i. The ring $\mathbb{Z}_p$ can thus be realized as the set of all sums of the type:

$$\sum_{i=0}^{\infty} a_i p^i \tag{1.2.4}$$

where $0 \leq a_i < p$ for each $i \geq 0$.

Suppose $x \in \mathbb{Q}_p$ does not satisfy $|x|_p \leq 1$. Then we can multiply x by a suitable power p^n with $n > 0$ so that $|p^n x|_p \leq 1$. It immediately follows that the field $\mathbb{Q}_p$, can thus be realized as the set of all sums of the type:

$$\sum_{i=k}^{\infty} a_i p^i \tag{1.2.5}$$

where $0 \leq a_i < p$ for each $i \geq k$ and $k \in \mathbb{Z}$ arbitrary. The actual mechanics of performing addition, subtraction, multiplication, and division in the field $\mathbb{Q}_p$ is very similar to what we do in the field $\mathbb{R}$ where every element is of the form

$$a_k 10^k + a_{k-1} 10^{k-1} + \cdots \tag{1.2.6}$$

with $0 \leq a_i \leq 9$ for all $i \geq k$. The main difference in $\mathbb{Q}_p$ is that the expansion

$$a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} \cdots$$

goes up instead of down as in (1.2.6).

Here is an example of multiplication in $\mathbb{Q}_5$. Note that the multiplication and carrying procedures mimic the case of multiplication in $\mathbb{R}$ except that we move from left to right instead of right to left.

$$\begin{array}{rl} & 2\cdot 5^{-1} + 4\cdot 5^{0} + 3\cdot 5^{1} + 2\cdot 5^{2} + \cdots \\ \times & 1\cdot 5^{-2} + 3\cdot 5^{-1} + 2\cdot 5^{0} + 1\cdot 5^{1} + \cdots \\ \hline & 2\cdot 5^{-3} + 4\cdot 5^{-2} + 3\cdot 5^{-1} + 2\cdot 5^{0} + \cdots \\ & \quad + 1\cdot 5^{-2} + 3\cdot 5^{-1} + 1\cdot 5^{0} + 3\cdot 5^{1} + \cdots \\ & \quad\quad + 4\cdot 5^{-1} + 3\cdot 5^{0} + 2\cdot 5^{1} + \cdots \\ & \quad\quad\quad + 2\cdot 5^{0} + 4\cdot 5^{1} + \cdots \\ \hline & 2\cdot 5^{-3} + 0\cdot 5^{-2} + 1\cdot 5^{-1} + 0\cdot 5^{0} + 1\cdot 5^{1} + \cdots. \end{array}$$

We give one more example of the type of infinite expansion that occurs in $\mathbb{Q}_p$ which is analogous to the expansion $\frac{1}{3} = 0.33333\ldots$ that occurs in $\mathbb{R}$.

Example 1.2.7 Let a be an integer coprime to the prime p. Let $f \geq 1$ be a fixed integer. Then there exist integers $\bar{a}, a_1, a_2, \ldots$ such that

$$\frac{1}{a} = \bar{a} + a_f p^f + a_{f+1} p^{f+1} + a_{f+2} p^{f+2} + \cdots \in \mathbb{Q}_p$$

where $a \cdot \bar{a} \equiv 1 \pmod{p^f}$ with $0 < \bar{a} < p^f$ and $0 \leq a_i < p$ for $i = f, f+1, f+2, \ldots$.

Since $|a^{-1}|_p = 1$ it follows that a^{-1} must be in $\mathbb{Z}_p$ and, thus, have an expansion of type (1.2.4). We require

$$a \cdot \left(\bar{a} + a_f p^f + a_{f+1} p^{f+1} + \cdots\right) = 1$$

from which it easily follows that $a\bar{a} \equiv 1 \pmod{p^f}$.

Note that p-adic expansions of p-adic numbers are always unique. This is not the case for decimal expansions of real numbers. For example: $1.000\ldots = 0.999\ldots$.

Algebraic construction of $\mathbb{Q}_p$: Let $A_1, A_2, A_3, \ldots$ be an infinite set of groups, rings, or fields. We assume that for every pair of positive integers i, j with $i > j$ there exists a homomorphism

$$f_{i,j} : A_i \to A_j. \tag{1.2.8}$$

Assume also that whenever i, j, k are positive integers satisfying $i > j > k$, that

$$f_{i,k} = f_{j,k} \circ f_{i,j}. \tag{1.2.9}$$

Definition 1.2.10 (Inverse limit) Let $A_1, A_2, A_3, \dots$ be an infinite set of groups, rings, or fields. Assume that for all positive integers $i > j$ that homomorphisms $f_{i,j}$ exist satisfying (1.2.8), (1.2.9). Then the inverse limit of the A_i, denoted

$$\varprojlim A_i$$

is defined to be the set of all infinite sequences $(a_1, a_2, a_3, \dots)$ where $a_i \in A_i$ for all $i \geq 1$ and $f_{i,j}(a_i) = a_j$ for all $i > j \geq 1$.

The inverse limit inherits the algebraic structure of the sets A_i. It will be either a group, ring or field.

In the algebraic approach to the construction of $\mathbb{Q}_p$ we first construct (using the inverse limit) the ring of p-adic integers, denoted $\mathbb{Z}_p$. The field $\mathbb{Q}_p$ is then constructed as the field of quotients of $\mathbb{Z}_p$, consisting of all elements of the form a/b with $a, b \in \mathbb{Z}_p$ and $b \neq 0$. Note that $\mathbb{Z}_p$ is an integral domain.

Let p be a prime and let i be a positive integer. Then the set

$$A_i := \left\{ a_0 + a_1 p + \cdots a_{i-1} p^{i-1} \;\middle|\; 0 \leq a_\ell < p \text{ for all } 0 \leq \ell < i \right\} \tag{1.2.11}$$

determines a finite ring with p^i elements which is canonically identified with the quotient ring $(\mathbb{Z}/p^i\mathbb{Z})$. The algebraic operations are addition and multiplication modulo p^i. For every $i > j$, we have the canonical homomorphism $f_{i,j} : A_i \to A_j$ defined by

$$f_{i,j}\left(a_0 + a_1 p + \cdots a_{i-1} p^{i-1}\right) = a_0 + a_1 p + \cdots a_{j-1} p^{j-1},$$

which simply drops off the tail end terms in the sum. It easily follows from Definition 1.2.10 that an element of the inverse limit is a sequence of the form

$$(a_0, \quad a_0 + a_1 p, \quad a_0 + a_1 p + a_2 p^2, \quad a_0 + a_1 p + a_2 p^2 + a_3 p^3, \quad \dots).$$

Formally, we define the infinite sum $\sum_{i=0}^{\infty} a_i p^i$ to be the sequence above. Then

$$\varprojlim \left(\mathbb{Z}/p^i\mathbb{Z}\right) = \left\{ \sum_{i=0}^{\infty} a_i p^i \;\middle|\; 0 \leq a_i < p \text{ for all } i \geq 0 \right\}. \tag{1.2.12}$$

Definition 1.2.13 (Ring of p-adic integers $\mathbb{Z}_p$) Let p be a prime number. The ring of p-adic integers $\mathbb{Z}_p$ is defined to be the inverse limit of finite rings given by (1.2.11).

1.3 Adeles and ideles over $\mathbb{Q}$

The completion of $\mathbb{Q}$ with respect to the archimedean absolute value $|\ |_\infty$ is just $\mathbb{R}$ which we also denote as $\mathbb{Q}_\infty$. Formally, the ring of adeles over $\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}$, is a ring determined by the restricted product (relative to the subgroups $\mathbb{Z}_p$)

$$\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_p \mathbb{Q}_p,$$

where restricted product (relative to the subgroups $\mathbb{Z}_p$) means that all but finitely many of the components in the product are in $\mathbb{Z}_p$.

Definition 1.3.1 (Adeles) The ring of adeles over $\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}$, is defined by

$$\mathbb{A}_\mathbb{Q} := \Big\{ \{x_\infty, x_2, x_3, \ \dots\} \ \Big|\ x_v \in \mathbb{Q}_v \,(\forall\, v \leq \infty),\ x_p \in \mathbb{Z}_p, \\ (\forall \text{ but finitely many } p) \Big\}.$$

Given two adeles

$$x = \{x_\infty,\ x_2,\ x_3,\ \dots\}, \qquad x' = \{x'_\infty,\ x'_2,\ x'_3,\ \dots\},$$

we define addition and multiplication (the ring operations) as follows

$$x + x' := \{x_\infty + x'_\infty,\ x_2 + x'_2,\ x_3 + x'_3,\ \dots\}$$
$$x \cdot x' := \{x_\infty \cdot x'_\infty,\ x_2 \cdot x'_2,\ x_3 \cdot x'_3,\ \dots\}.$$

Recall that a topological space X is called locally compact if every point of X has a compact neighborhood. For example, $\mathbb{Q}_p$ is locally compact and $\mathbb{Z}_p$ is compact. Furthermore, $\mathbb{A}_\mathbb{Q}$ can be made into a locally compact topological ring by taking as a basis for the topology all sets of the form

$$U \times \prod_{p \notin S} \mathbb{Z}_p$$

where S is any finite set of primes containing ∞, and U is any open subset in the product topology on the finite product $\prod_{v \in S} \mathbb{Q}_v$. (This follows the Tychonoff theorem, see [Munkres, 1975].)

The ideles of $\mathbb{Q}$ are defined to be the multiplicative subgroup of $\mathbb{A}_\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}^\times$.

Definition 1.3.2 (Ideles) The multiplicative group of ideles over $\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}^\times$, is defined by

$$\mathbb{A}_\mathbb{Q}^\times := \Big\{ \{x_\infty, x_2, \ \dots\} \in \mathbb{A}_\mathbb{Q} \ \Big|\ x_v \in \mathbb{Q}_v^\times \,(\forall v),\ x_p \in \mathbb{Z}_p^\times, \\ (\forall \text{ but finitely many } p) \Big\}.$$

Here $\mathbb{Z}_p^\times$ denotes the multiplicative group of units of $\mathbb{Z}_p$. Clearly, $u \in \mathbb{Z}_p^\times$ if and only if $|u|_p = 1$. The ideles over $\mathbb{Q}$ also form a locally compact topological group with the basis of the topology consisting of the open sets

$$U \times \prod_{p \notin S} \mathbb{Z}_p^\times$$

where U is an open set in $\prod_{v \in S} \mathbb{Q}_v^\times$ and S is any finite set of primes containing ∞. Here, the topology on the finite product $\prod_{v \in S} \mathbb{Q}_v^\times$ is the product topology.

Warning: The topology of the ideles is not the topology induced from the adeles. It is quite different.

Definition 1.3.3 (Finite adeles) The ring of finite adeles over $\mathbb{Q}$, denoted $\mathbb{A}_{\text{finite}}$, is defined by

$$\mathbb{A}_{\text{finite}} := \Big\{ \{x_2, x_3, \ \dots\} \ \Big| \ x_p \in \mathbb{Q}_p \ (\forall\ p < \infty),\ x_p \in \mathbb{Z}_p\,, \\ (\forall \text{ but finitely many } p) \Big\}.$$

There is a natural embedding of $\mathbb{A}_{\text{finite}}$ into $\mathbb{A}_{\mathbb{Q}}$ given by

$$\{x_2, x_3, \ \dots\} \mapsto \{0, x_2, x_3, \ \dots\}.$$

Definition 1.3.4 (Finite ideles) The group of finite ideles over $\mathbb{Q}$, denoted $\mathbb{A}_{\text{finite}}^\times$, is defined by

$$\mathbb{A}_{\text{finite}}^\times := \Big\{ \{x_2, x_3, \ \dots\} \ \Big| \ x_p \in \mathbb{Q}_p^\times \ (\forall\ p < \infty),\ x_p \in \mathbb{Z}_p^\times\,, \\ (\forall \text{ but finitely many } p) \Big\}.$$

There is a natural embedding of $\mathbb{A}_{\text{finite}}^\times$ into $\mathbb{A}_{\mathbb{Q}}^\times$ given by

$$\{x_2, x_3, \ \dots\} \mapsto \{1, x_2, x_3, \ \dots\}.$$

1.4 Action of $\mathbb{Q}$ on the adeles and ideles

The ring $\mathbb{Q}$ can be embedded in the adeles as follows. It is clear that for any fixed $q \in \mathbb{Q}$ that $|q|_v > 1$ for only finitely many $v \leq \infty$. Thus q lies in $\mathbb{Z}_p$ for all but finitely many $p < \infty$.

Let $q \in \mathbb{Q}$. Then

$$\{q, q, q, \ \dots\} \in \mathbb{A}_{\mathbb{Q}}.$$

This is usually referred to as a diagonal embedding. It follows that $\mathbb{Q}$ may be considered as a subring of $\mathbb{A}_{\mathbb{Q}}$. Viewing $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{Q}$ as additive groups, it is

then natural to take the quotient $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$. Another way to view this quotient is to define an additive action (denoted +) of $\mathbb{Q}$ on $\mathbb{A}_{\mathbb{Q}}$ by the formula

$$q + x := \{q + x_\infty,\ q + x_2,\ q + x_3,\ \ldots\}$$

for all $x = \{x_\infty, x_2, x_3, \ldots\} \in \mathbb{A}_{\mathbb{Q}}$ and all $q \in \mathbb{Q}$. Here $q + x_v$ denotes addition in $\mathbb{Q}_v$. This is a continuous action and $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}_{\mathbb{Q}}$ in the sense that for each $q \in \mathbb{Q}$, there is a subset $U \subset \mathbb{A}_{\mathbb{Q}}$, which is open in the topology on $\mathbb{A}_{\mathbb{Q}}$, such that $U \cap \mathbb{Q} = \{q\}$.

We now introduce the notion of a fundamental domain for the action of an arbitrary group on an arbitrary set X.

Definition 1.4.1 (Fundamental domain) Let a group G act on a set X (on the left). A fundamental domain for this action is a subset $D \subset X$ which satisfies the following two properties:

(1) For each $x \in X$, there exists $d \in D$ and $g \in G$ such that $gx = d$.
(2) The choice of d in (1) is unique.

Remarks A fundamental domain is precisely a choice of one point from each orbit of G. If $G\backslash X$ is the quotient space with the quotient topology and $\pi : X \to G\backslash X$ is the quotient map, then the fundamental domain is the image of a section $\sigma : G\backslash X \to X$. (This is a set theoretic section, it need not be continuous.)

The construction of an explicit fundamental domain for the action of the additive group $\mathbb{Q}$ on the adele group $\mathbb{A}_{\mathbb{Q}}$ is equivalent to a generalization of the ancient Chinese remainder theorem.

Theorem 1.4.2 (Chinese Remainder Theorem) *Let $p_1, p_2, \ldots p_n$ be distinct primes. Let $e_1, e_2, \ldots, e_n$ be positive integers and $c_1, c_2, \ldots, c_n$ be arbitrary integers. Then the system of linear congruences*

$$\begin{aligned} x &\equiv c_1 \pmod{p_1^{e_1}} \\ x &\equiv c_2 \pmod{p_2^{e_2}} \\ &\vdots \\ x &\equiv c_n \pmod{p_n^{e_n}} \end{aligned}$$

has a unique solution $x \pmod{p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}}$.

Proof A simple proof can be obtained by explicitly constructing a solution to the system of linear congruences. Set $N = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. For each $1 \le i \le n$ define an integer u_i by the condition

$$\frac{N}{p_i^{e_i}} \cdot u_i \equiv 1 \pmod{p_i^{e_i}}.$$

Then one easily checks that the element

$$x \equiv c_1 \frac{N}{p_1^{e_1}} \cdot u_1 + c_2 \frac{N}{p_2^{e_2}} \cdot u_2 + \quad \cdots \quad + c_n \frac{N}{p_n^{e_n}} \cdot u_n$$

satisfies $x \equiv c_i \pmod{p_i^{e_i}}$ for all $1 \leq i \leq n$. We leave the proof of uniqueness to the reader. □

Example 1.4.3 Consider the system of linear congruences

$$x \equiv 2 \pmod{3^2}$$
$$x \equiv 1 \pmod{5^3}$$
$$x \equiv 3 \pmod{7}.$$

Then u_1 is defined by the congruence $5^3 \cdot 7 \cdot u_1 \equiv 1 \pmod{3^2}$, and $u_1 = 5$. Similarly, $3^2 \cdot 7 \cdot u_2 \equiv 1 \pmod{5^3}$ and $u_2 = 2$, while $3^2 \cdot 5^3 \cdot u_3 \equiv 1 \pmod{7}$ and $u_3 = 3$. It follows that

$$x \equiv 2 \cdot 5^3 \cdot 7 \cdot 5 + 3^2 \cdot 7 \cdot 2 + 3 \cdot 3^2 \cdot 5^3 \cdot 3 \equiv 3251 \pmod{3^2 \cdot 5^3 \cdot 7}.$$

A modern version of the Chinese Remainder Theorem (Theorem 1.4.2) can be given in terms of p-adic absolute values.

Theorem 1.4.4 (Weak approximation) *Let $p_1, p_2, \ldots, p_n$ be distinct primes. Let $c_i \in \mathbb{Q}_{p_i}$ for each $i = 1, 2, \ldots, n$. Then for every $\epsilon > 0$, there exists an $\alpha \in \mathbb{Q}$ such that*

$$|\alpha - c_i|_{p_i} < \epsilon$$

for all $1 \leq i \leq n$. Furthermore, α may be chosen so that the denominator, when written in lowest terms, is not divisible by any primes other than $p_1, \ldots, p_n$.

Proof The general case follows easily from the case when $c_i \in \mathbb{Z}_{p_i}$ for all i. As $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we may then replace c_i by $c_i' \in \mathbb{Z}$. At this point the statement reduces to the classical form, given in Theorem 1.4.2. □

Proposition 1.4.5 (Strong approximation for adeles) *A fundamental domain D for $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ is given by*

$$D = \left\{ \{x_\infty, x_2, x_3, \ \ldots\} \;\middle|\; 0 \leq x_\infty < 1, \ x_p \in \mathbb{Z}_p \text{ for all finite primes } p \right\}$$
$$= [0, 1) \cdot \prod_p \mathbb{Z}_p.$$

That is, we have

$$\mathbb{A}_{\mathbb{Q}} = \bigcup_{\beta \in \mathbb{Q}} \{\beta + D\}, \qquad (\textit{disjoint union}).$$