This book provides a comprehensive presentation of geometric results, primarily from the theory of convex sets, that have been proved by the use of Fourier series or spherical harmonics. Almost all these geometric results appear here in book form for the first time.

An important feature of the book is that all necessary tools from the classical theory of spherical harmonics are presented with full proofs. These tools are used to prove geometric inequalities, stability results, uniqueness results for projections and intersections by hyperplanes or half-spaces, and characterizations of rotors in convex polytopes. Again, full proofs are given. To make the treatment as self-contained as possible the book begins with background material in analysis and the geometry of convex sets.

This treatise will be welcomed both as an introduction to the subject and as a reference book for pure and applied mathematicians.

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To Helga

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PREFACE

In 1901 Adolf Hurwitz published a short note showing that Fourier series can be used to prove the isoperimetric inequality for domains in the Euclidean plane, and in a subsequent article he showed how spherical harmonics can be utilized to prove an analogous inequality for three-dimensional convex bodies. A few years later Hermann Minkowski used spherical harmonics to prove an interesting characterization of (three-dimensional) convex bodies of constant width. The work of Hurwitz and Minkowski has convincingly shown that a study of this interplay of analysis and geometry, in particular of Fourier series and spherical harmonics on the one hand, and the theory of convex bodies on the other hand, can lead to interesting geometric results. Since then many articles have appeared that explored the possibilities of such methods.

The aim of the present book is to provide a fairly comprehensive exposition of geometric results, more specifically, of results in the theory of convex sets, that have been proved by the use of Fourier series or spherical harmonics. Almost all theorems that are stated are also proved. Furthermore, to make the book more self-contained, all results from the theory of spherical harmonics that are used are also proved. Thus the only prerequisite for reading this book is some familiarity with the basic facts of the theory of (finite dimensional) convex sets and the theory of functions of real variables.

The book consists of five chapters. The first two of these contain preparatory material from analysis and the geometry of convex sets. These topics are reviewed to establish consistent notation and to formulate some known results for later reference. Moreover, some auxiliary results that are not part of the standard textbook literature are formulated and proved.

In Chapter 3 the theory of spherical harmonics is developed to the extent that it is useful for later geometric applications. An attempt has been made to present this material at the level of classical analysis, since a more abstract approach would not have significantly enlarged the area of geometric applications, but could have made it more difficult for some readers to acquaint themselves with the necessary analytic tools.

Chapters 4 and 5 contain the geometric applications. Readers familiar with the theory of convex sets and spherical harmonics may start immediately with these chapters, using the previous material only to acquaint themselves with the notation, definitions, and some auxiliary results. Chapter 4 deals with applications of Fourier series and consequently with geometric results in the two-dimensional Euclidean space. Some of these results are discussed since higher dimensional analogues are not available, others because they provide good examples of the method of proof that will be used later in the more complicated higher dimensional situation. Chapter 5 can be considered the principal part of the book. It deals with applications of spherical harmonics to geometric problems in Euclidean space without dimension limitations.

Some of the results presented here are proved in essentially the same way as in the articles where they originally appeared; in other cases the proofs have been modified considerably or even replaced by new ones if this was deemed to benefit the overall presentation of the subject. Several theorems too far removed from the central area of this book or better proved by other methods are mentioned in a more casual manner. This is usually done at the end of each section in a kind of appendix headed "Remarks and References." In these paragraphs one also finds the pertinent references to the original literature and some historical comments. No attempt has been made, however, to present a comprehensive survey of the long and ramified history of spherical harmonics and their applications.

Several mathematicians were kind enough to read at least some parts of earlier versions of this book and to make valuable comments and suggestions. In particular, I wish to thank Professors Gulbank D. Chakerian, Paul R. Goodey, Richard J. Gardner, Peter M. Gruber, Erwin Lutwak, Krzysztof Przesławski, and Rolf Schneider. I would like to express my special gratitude to Professor Erwin Lutwak, who originally suggested that I write a book for the Encyclopedia series, and to Professor Rolf Schneider, whose extensive contributions and interesting results in the subject area of this book have stimulated me to study this topic and to write about it.

Remarks on Matters of Presentation and Notation

Definitions are stated in the conventional way with "if" instead of "if and only if." The concepts of lemmas, theorems, and corollaries are used with their well-established meanings, but there also appear "propositions." These are meant to be either auxiliary results having some independent interest or significance, or results of a less important or more isolated kind than those listed as theorems.

All lemmas, theorems, corollaries, propositions, and some formulas are marked by three numbers. The first one indicates the chapter, the second number refers to the section within the chapter, and the third one indicates the position within the section. References are given by the name of the author (or the names of the authors) in small capital letters followed by the year of publication, sometimes marked with a letter if there are several articles by the same author within one year. Depending on the context, when such a reference is given it may mean either the author or the publication. For example a phrase like "see MINKOWSKI (1903)" means obviously that one should see the article which Minkowski published in 1903 and which is listed as such in the References at the end of the book. On the other hand a statement like "this was proved by MINKOWSKI (1903)" clearly refers to the person and the year when the result was published. If an article or a book has more than one publication date the earliest one is used for reference; if it has not yet appeared in print it is quoted as "in press," possibly followed by a letter if there is more than one such prospective publication by the same author.

The word "function" without any further information on its range is always meant to be a real valued function. Similarly, the noun "constant" without any further specification is used to indicate a real valued constant. We sometimes employ such convenient and customary (but mildly contradictory) phrases as "c_n is a constant depending on n only," which, strictly speaking, means that c_n is a function of the single variable n.

There is a list of frequently used symbols at the end of the book (after the References). Of particular importance is that, except for its use in integrals and derivatives, the letter d will always denote the dimension of the space under consideration. Unless specified otherwise, it is assumed that $d \geq 2$. It also should be observed that κ_d and σ_d are used exclusively to denote, respectively, the volume and surface area of a d-dimensional unit ball. The term "measure" always means a nonnegative measure; if negative values are permitted the corresponding concept will be called a signed measure.