This book gives a detailed account of recent work on relations between commutative algebra and intersection theory, with a particular emphasis on applications of the theory of local Chern characters. This theory is the result of many years of development, having originated in topology and been introduced in algebraic geometry about thirty years ago. Building on the algebraic form described in *Intersection Theory* by W. Fulton, Paul Roberts presents further developments and important algebraic applications that were not known at the time Fulton's book was written. Some of these applications come from the author's own work. He also discusses the background in commutative algebra and related questions in homological algebra and describes the relations between these subjects, including extensive discussions of the homological conjectures and of the use of the Frobenius map.

Students and researchers specializing in commutative algebra will find access to a wide range of new ideas in this book.

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133 Multiplicities and Chern Classes in Local Algebra

PAUL C. ROBERTS
University of Utah

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Preface

The study of multiplicities has been of major importance in commutative algebra since the beginnings of the subject. It developed from the notion of the multiplicity of a root of a polynomial and from counting multiplicities of intersections in algebraic geometry, and it has influenced the development of the subject in many ways. On the other hand, the notion of Chern classes was originally developed in topology, and it is comparatively recently that it has been used in algebraic geometry and even more recently that it has made its way into commutative algebra. It is the aim of this book to present the theories of multiplicities and of Chern classes in an algebraic setting and to describe their relations with each other and with other topics in the field.

There are two somewhat different notions of multiplicities that will be discussed at length in this book. They both originate from algebraic geometry but from somewhat different sources. The first notion of multiplicity that we consider comes from the multiplicity of a variety at a point. This is best illustrated by an example. Suppose we have a curve in a plane that crosses itself, perhaps more than once, at a point. It then makes sense to take the multiplicity of the curve at the point to be the number of components locally at the point. But this straightforward definition in this case already raises problems: For example, if the curve is tangent to itself, it is not immediately clear how the multiplicity should be counted. In addition, the more complicated possibilities in higher dimension make a general definition less obvious. A good notion of multiplicity in this sense was introduced by Samuel, and we will refer to it as *Samuel multiplicity*. Since we will mostly be concerned with algebraic notions here, we will define multiplicities for local rings; in the geometric situation above, the local ring would be the local ring of functions at the point. We also discuss generalizations of this concept to ideals, modules, and sets of ideals.

The second notion is that of the multiplicity of intersection of two subvarieties of a variety. This can be considered a direct generalization of the multiplicity of roots of a polynomial, which is the case of the intersection of the graph of the polynomial with the x-axis. The question of defining intersection multiplicities algebraically in general has been a central topic in algebraic geometry for many years, and several definitions have been proposed. The one we consider here was introduced by Serre, and it involves extensive use of homological algebra. This homological definition has led to a considerable number of purely algebraic questions and conjectures relating to properties of modules and complexes of free modules, and particularly to properties of modules of finite projective dimension.

The first part of this book deals with the various concepts of multiplicity, as well as related ideas in the field of commutative algebra. We prove some of the basic theorems in this field, including properties of prime ideals and dimension, which can be found in the fundamental books on the subject such as Matsumura's *Commutative Rings* or the *Introduction to Commutative Algebra* of Atiyah and Macdonald. On the other hand, we assume that the reader has a knowledge of the basic results that can be found in these or similar books. The elementary results that we prove in this book are those closely related to the rest of the theory; we give references for others.

The other topics we cover revolve around the so-called homological conjectures, which arose in part from questions on multiplicities. We include sections on homological algebra, Cohen-Macaulay properties, Koszul complexes, and dualizing complexes and discuss their relations to questions on multiplicities and to these conjectures. We also discuss some of the methods that have been used to solve these problems. Some of these involve Cohen-Macaulay modules and duality and are presented in the first five chapters. We also discuss the use of the Frobenius map for rings of positive characteristic, which was introduced into this subject by Peskine and Szpiro to answer some of these questions. The Frobenius map is discussed in Chapter 7 after a discussion of the homological conjectures in Chapter 6.

Another important method was made possible by the development of theory of local Chern characters of Baum, Fulton, and MacPherson [5]. This made it possible to solve some of the problems in mixed characteristic, and it also sheds new light on the theory of intersection multiplicities and of Samuel multiplicities. Most of this theory is worked out in a geometric setting in Fulton [17], which is a source of many of the results in this book.

The main aim of the second half of the book is to present the algebraic theory of local Chern characters in a self-contained manner. Although it is not possible to make it completely self-contained, this presentation comes quite close, and the necessary results for proving the properties needed in the applications are proven in detail. In addition, we have consistently emphasized algebraic methods in the constructions.

The background includes the theory of Chow group, which is introduced in the first chapter together with basic properties of prime ideals, functorial properties of projective maps, the basic theory of Chern classes of locally free sheaves, and some of the fundamental properties of the Grassmannian. In some cases our presentation is not totally standard; in particular, projective schemes are defined for multigraded rings, which arise naturally in this situation and which are related to the theory of Hilbert polynomials and multiplicities presented in Chapter 2. In addition, coherent sheaves on projective schemes are almost exclusively treated in terms of the graded modules that are associated to them. In any case, the proofs of theorems are essentially the same, as they usually come down to the case of ordinary projective schemes defined by graded rings.

I would like to thank the students in a course I gave in these topics at the University of Utah last year, using preliminary versions of part of this material. In particular, I want to thank Ionut Ciocan-Fontanine, Elizabeth Jones, Saule Zhoshina, Chin-yi Chan, and Sean Sather-Wagstaff, who made numerous suggestions and contributions to the readability and correctness of the book. And finally, I want to give special thanks to my wife Anne, whose unending patience was essential in bringing this project to completion.