# Part One

## Multiplicities

# 1

# Prime Ideals and the Chow Group

The main purpose of this chapter is to define the Chow group of a Noetherian ring and prove several of its basic properties. The Chow group is a quotient of the free group whose generators are the prime ideals of the ring, and we devote the first section to a summary of some of the main properties of prime ideals, and in particular the prime ideals associated to a finitely generated module. Much of the material in the first section can be found in several books on commutative algebra, such as Matsumura [44] or Atiyah and Macdonald [1]. While we prove some of these basic facts, we use others without proof, giving a reference to a place in one of these books where a proof can be found. Most of the results of Section 2, in a more general setting, can be found in Fulton [17].

## 1.1 Prime Ideals in Noetherian Rings

All rings will be assumed to be commutative and have an identity element. A module $M$ over a commutative ring $A$ is *Noetherian* if it has the ascending chain condition on submodules, or, equivalently, if every submodule is finitely generated. The ring $A$ is Noetherian if it is Noetherian as an $A$-module, which means that it has the ascending chain condition on ideals or that every ideal is finitely generated. We recall that every finitely generated module over a Noetherian ring is Noetherian.

Unless specifically stated otherwise, all rings in this book will be assumed to be Noetherian. By the Hilbert basis theorem (see [1, Theorem 7.5]), this class of rings includes all rings that are finitely generated as algebras over a field or over the ring of integers, and the class is closed under the operations of localization and completion.

Let $A$ be a ring. An ideal $\mathfrak{p}$ of $A$ is *prime* if $\mathfrak{p} \neq A$ and if whenever $ab \in \mathfrak{p}$, we have either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The zero ideal of $A$ is prime if and only if $A$

is an integral domain. While this definition is valid in any commutative ring, whether Noetherian or not, the set of prime ideals in Noetherian rings has many special properties. We denote the set of prime ideals of $A$ by $\text{Spec}(A)$.

An ideal $\mathfrak{a}$ of $A$ is an *annihilator ideal* for the module $M$ if there exists a nonzero element $m \in M$ such that $\mathfrak{a} = \{x \in A \mid xm = 0\}$. The next proposition shows the existence of prime ideals that are annihilator ideals.

**Proposition 1.1.1** *Let $M$ be an $A$-module. Then every ideal that is maximal in the set of annihilator ideals for $M$ is prime.*

*Proof* Let $\mathfrak{a}$ be a maximal annihilator ideal for $M$, and suppose that $\mathfrak{a}$ is the annihilator of the element $m$ in $M$. Let $a$ and $b$ be elements of $A$ such that neither $a$ nor $b$ is in $\mathfrak{a}$. Since $a$ is not in $\mathfrak{a}$, and $\mathfrak{a}$ is the annihilator of $m$, we have $am \neq 0$. On the other hand, $\mathfrak{a}$ annihilates $am$, so, by the maximality of $\mathfrak{a}$, we must have that $\mathfrak{a}$ is equal to the annihilator of $am$. Since $b \notin \mathfrak{a}$, this means that $abm \neq 0$, so $ab \notin \mathfrak{a}$. Thus, $\mathfrak{a}$ is prime. □

In general there is no reason for maximal annihilator ideals to exist (they do not satisfy the conditions of Zorn's lemma, for example), but if the ring is Noetherian the ascending chain condition implies that every annihilator ideal is contained in a maximal annihilator ideal and hence in a prime annihilator ideal.

Let $A$ be a Noetherian ring, and let $M$ be a finitely generated $A$-module. We say that a prime ideal $\mathfrak{p}$ is *associated* to $M$, or is an associated prime of $M$, if there is an element of $M$ whose annihilator is exactly $\mathfrak{p}$, or, equivalently, if there is an embedding of $A/\mathfrak{p}$ into $M$. Proposition 1.1.1 implies that every nonzero module has at least one associated prime. In fact, it shows that the union of all associated primes of $M$ is exactly the set of zero-divisors on $M$, where a zero-divisor is an element $a \in A$ such that there exists a nonzero element $m \in M$ with $am = 0$. In fact, every associated prime ideal consists of zero-divisors, and if $a$ is a zero-divisor, then it is contained in the annihilator of a nonzero element and hence by Proposition 1.1.1 in an associated prime ideal.

The *support* of $M$ is the set of prime ideals $\mathfrak{p}$ for which the localization $M_{\mathfrak{p}}$ is not zero. The only associated prime ideal of $A/\mathfrak{p}$ is $\mathfrak{p}$, and the support of $A/\mathfrak{p}$ consists of all primes containing $\mathfrak{p}$. The localization of $A/\mathfrak{p}$ at $\mathfrak{p}$ is the quotient field of the integral domain $A/\mathfrak{p}$, which we denote $k(\mathfrak{p})$. For any module $M$, every associated prime ideal of $M$ is in the support of $M$.

Let $S$ be a multiplicatively closed subset of $A$, and let $A_S$ and $M_S$ denote the localizations of $A$ and $M$ at $S$. Every prime ideal $\mathfrak{p}$ generates an ideal $\mathfrak{p}_S$ of $A_S$, and it is not hard to check that $\mathfrak{p}_S$ is prime if $\mathfrak{p}$ does not meet $S$.

We summarize the behavior of prime ideals under localization in the following proposition.

**Proposition 1.1.2** *Let $M$ be an $A$-module as above, and let $S$ be a multiplicatively closed subset of $A$. Then*

(i) *The map that sends $\mathfrak{p}$ to $\mathfrak{p}_S$ defines a one-one correspondence between the set of prime ideals of $A$ that do not meet $S$ and the set of prime ideals of $A_S$.*
(ii) *The support of $M_S$ is the subset of* $\operatorname{Spec}(A_S)$ *consisting of those $\mathfrak{p}_S$ for which $\mathfrak{p}$ is in the support of $M$.*
(iii) *The set of associated primes of $M_S$ is the subset of* $\operatorname{Spec}(A_S)$ *consisting of those $\mathfrak{p}_S$ for which $\mathfrak{p}$ is associated to $M$.*

*Proof* We note first that $\mathfrak{p}_S = A_S$ if and only if $\mathfrak{p} \cap S$ is not empty. Assume that $\mathfrak{p} \cap S = \emptyset$. Then if $a/s \in \mathfrak{p}_S$, we have $ta \in \mathfrak{p}$ for some $t \in S$, and since $t$ cannot be in $\mathfrak{p}$ and $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus, $a/s$ is in $\mathfrak{p}_S$ if and only if $a \in \mathfrak{p}$.

Let $f$ be the map from $A$ to $A_S$ that sends $a$ to $a/1$. If $\mathfrak{q} \in \operatorname{Spec}(A_S)$, then $\mathfrak{q} = (f^{-1}(\mathfrak{q}))_S$. But we just proved that if $\mathfrak{p}$ is a prime ideal in $A$, then $\mathfrak{p} = f^{-1}(\mathfrak{p}_S)$, so it now follows that the map that sends $\mathfrak{q}$ to $f^{-1}(\mathfrak{q})$ is an inverse to the map that sends $\mathfrak{p}$ to $\mathfrak{p}_S$.

The second statement follows from the fact that if $\mathfrak{p} \cap S = \emptyset$, then $(M_S)_{\mathfrak{p}_S} \cong M_{\mathfrak{p}}$.

If $\mathfrak{p}$ is an associated prime of $M$, then we have an embedding of $A/\mathfrak{p}$ into $M$, and localizing gives an embedding of $(A/\mathfrak{p})_S$ into $M_S$, so $\mathfrak{p}_S$ is an associated prime of $M_S$. Conversely, suppose that $\mathfrak{p}_S$ is an associated prime of $M_S$ and is the annihilator of $m/s$. By multiplying by the unit $s$, we may assume that $s = 1$. The annihilator of $m$ will then be contained in $\mathfrak{p}$, and we claim that some multiple $tm$ will have annihilator exactly $\mathfrak{p}$. In fact, if $x \in \mathfrak{p}$, then $(x/1)(m/1) = xm/1 = 0$ in $M_S$, so $tmx = 0$ for some $t \in S$, and thus $x$ annihilates $tm$. Since $\mathfrak{p}$ is finitely generated, we may find a $t$ such that every element of the ideal $\mathfrak{p}$ annihilates $tm$. Since the annihilator of $tm$ is still contained in $\mathfrak{p}$, we have that $\mathfrak{p}$ is the annihilator of $tm$ and is an associated prime of $M$. □

An important consequence of Proposition 1.1.2 is that every prime ideal that is minimal in the support of $M$ is associated to $M$. In fact, if $\mathfrak{p}$ is minimal in the support of $M$, then $\mathfrak{p}_{\mathfrak{p}}$ is the only prime ideal in the support of $M_{\mathfrak{p}}$ over the localization $A_{\mathfrak{p}}$, so $\mathfrak{p}_{\mathfrak{p}}$ must be associated to $M_{\mathfrak{p}}$, and Proposition 1.1.2 implies that $\mathfrak{p}$ is associated to $M$.

The basic theorem on prime ideals associated to modules is the following.

**Proposition 1.1.3** *If*

$$0 \to M' \to M \to M'' \to 0$$

*is a short exact sequence, then*

(i) *The support of $M$ is the union of the supports of $M'$ and of $M''$.*
(ii) *Any associated prime of $M$ is an associated prime of $M'$ or of $M''$.*

*Proof* The first statement follows from the exactness of localization.

If $\mathfrak{p}$ is an associated prime of $M$, let $m$ be an element of $M$ whose annihilator is $\mathfrak{p}$. If the submodule generated by $m$ meets $M'$ only in 0, then $\mathfrak{p}$ is associated to $M''$. If not, there is a nonzero element in $Am \cap M'$, and its annihilator must be $\mathfrak{p}$, so $\mathfrak{p}$ is associated to $M'$. □

The next theorem is one of the main tools for studying modules over Noetherian rings.

**Theorem 1.1.4** *Let $M$ be a finitely generated module. There is a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

*of $M$ such that*

$$M_i/M_{i-1} \cong A/\mathfrak{p}_i$$

*for some prime ideal $\mathfrak{p}_i$ for each $i = 1, \ldots n$. The set of prime ideals $\mathfrak{p}_i$ thus obtained contains all the associated primes of $M$. In particular, the set of associated primes of $M$ is finite.*

*Proof* Proposition 1.1.1 implies that there exists a prime ideal $\mathfrak{p}_1$ and an embedding of $A/\mathfrak{p}_1$ into $M$. Let $M_1$ be the image of $A/\mathfrak{p}_1$ in $M$. This gives the first module in the filtration. We may then apply the same process to $M/M_1$ to get $M_2, \ldots M_n$ together with prime ideals $\mathfrak{p}_2, \ldots \mathfrak{p}_n$. Since $M$ is a Noetherian module this process must eventually stop, and we have a filtration of the required type.

To show that the set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ contains every associated prime of $M$, we use induction on $n$. If $n = 1$, $M \cong A/\mathfrak{p}_1$, and the only associated prime ideal of $A/\mathfrak{p}_1$ is $\mathfrak{p}_1$. If $n > 1$, we apply Proposition 1.1.3 to the short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0.$$

Every associated prime of $M$ must be associated to $M_1$ or $M/M_1$, and hence by induction it must be one of the $\mathfrak{p}_i$. □

One of the uses of this theorem is to show the existence of elements that are not zero-divisors on a module. We define a ring to be *local* if it has a unique maximal ideal. Let $A$ be a local ring, and let $M$ be a finitely generated $A$-module. If the maximal ideal $\mathfrak{m}$ of $A$ consists of zero-divisors on $M$, then it must be the union of the finite set of associated primes of $M$. The next proposition shows that then $\mathfrak{m}$ must be an associated prime ideal of $M$.

**Proposition 1.1.5** *If an ideal $\mathfrak{a}$ is contained in a finite union of prime ideals $\mathfrak{p}_1, \ldots \mathfrak{p}_n$, then we have $\mathfrak{a} \subset \mathfrak{p}_i$ for some $i$.*

*Proof* We prove this result by induction on $n$; if $n = 1$ it is clear. If $n > 1$, we may assume that we have no containment relations $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for $i \neq j$. Suppose that $\mathfrak{a} \not\subset \mathfrak{p}_i$ for all $i$. By induction we can find, for each $i$, an element $x_i \in \mathfrak{a}$ with $x_i \notin \mathfrak{p}_j$ for all $j \neq i$; since we are assuming that $\mathfrak{a}$ is contained in $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, we must have $x_i \in \mathfrak{p}_i$. Let $y_i = \prod_{j \neq i} x_j$ for each $i$. Then $y_i$ is in all the $\mathfrak{p}_j$ except $\mathfrak{p}_i$, and $\sum y_i$ is in $\mathfrak{a}$ but in none of the $\mathfrak{p}_i$. □

We state explicitly the consequence mentioned above for the maximal ideal of a local ring.

**Corollary 1.1.6** *Let $A$ be a local ring with maximal ideal $\mathfrak{m}$, and let $M$ be a finitely generated $A$-module. Then exactly one of the following holds:*

(i) *The ideal $\mathfrak{m}$ is an associated prime of $M$.*
(ii) *There exists an element $x \in \mathfrak{m}$ that is not a zero-divisor on $M$.*

*Proof* If $\mathfrak{m}$ is associated to $M$, then clearly every element of $\mathfrak{m}$ is a zero-divisor on $M$. On the other hand, if $\mathfrak{m}$ consists of zero-divisors on $M$, it is contained in the union of associated primes of $M$. By Proposition 1.1.5 it must therefore be contained in one of the associated primes, and since $\mathfrak{m}$ is maximal it must itself be an associated prime of $M$. □

## 1.2 Cycles and the Chow Group

Let $A$ be a Noetherian ring. The *dimension* (or Krull dimension) of $A$, denoted $\dim(A)$, is the supremum of integers $i$ for which there exists a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_i$ such that $\mathfrak{p}_j$ is properly contained in $\mathfrak{p}_{j+1}$ for each

$j$. For any module $M$ the dimension of $M$ is the supremum of lengths of chains of prime ideals in the support of $M$. In most cases we will consider, the dimension of a module or ring will be finite, but this is not true in general for Noetherian rings (see Nagata [48, Appendix A]). The dimension of a local ring in particular is finite; we will prove this statement in the next chapter. We note that the dimension of any quotient $A/\mathfrak{a}$ of $A$ is the same whether considered as a ring or as an $A$-module, since the support of $A/\mathfrak{a}$ is equal to $\text{Spec}(A/\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

If $\mathfrak{p}$ is a prime ideal of $A$, the *height* of $\mathfrak{p}$ is the dimension of the localization $A_{\mathfrak{p}}$, or, equivalently, the supremum of lengths of chains of prime ideals contained in $\mathfrak{p}$.

Some of the basic definitions in this chapter make sense for all Noetherian rings, but for much of the theory to work properly it will be necessary to set further restrictions. One property of dimension that will be necessary is the following: if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals such that $\mathfrak{p}$ is properly contained in $\mathfrak{q}$ and such that there are no prime ideals properly between $\mathfrak{p}$ and $\mathfrak{q}$, then $\dim(A/\mathfrak{p}) = \dim(A/\mathfrak{q}) + 1$. This condition is satisfied, for example, for finitely generated algebras over a field and for many other important cases, but it is not true in general even for localizations of finitely generated algebras over a field. We discuss the question of dimension more thoroughly in Chapter 4 and give a precise statement of the class of rings that we consider. We will temporarily assume that prime ideals satisfy the above condition, as they do in many important cases.

Let $X$ denote the set $\text{Spec}(A)$ of prime ideals of $A$. For each nonnegative integer $i$ we let $Z_i(X)$, or $Z_i(A)$, be the free Abelian group with basis consisting of all prime ideals $\mathfrak{p}$ such that the dimension of $A/\mathfrak{p}$ is $i$. We will on occasion refer to the dimension of $A/\mathfrak{p}$ as the dimension of $\mathfrak{p}$. The generator of $Z_i(X)$ corresponding to $\mathfrak{p}$ will be denoted $[A/\mathfrak{p}]$. The group $Z_i(X)$ is called the group of *cycles of $X$ of dimension $i$*. The direct sum of $Z_i(X)$ over all $i$ is called the group of cycles of $A$ and is denoted $Z_*(X)$ or $Z_*(A)$.

As an example, we take $A = k[X, Y]$, a polynomial ring in two variables. Since $k[X, Y]$ has dimension 2, $Z_k(A)$ is zero for all $k$ except for $k = 0, 1$, and 2. $Z_0(A)$ consists of the free Abelian group on the set of maximal ideals of $A$, and if $k$ is algebraically closed it follows from the Hilbert nullstellensatz (see Matsumura [44, §5]) that this set corresponds to the set of points $(a, b)$ in $k \times k$. $Z_1(A)$ has a basis consisting of primes $\mathfrak{p}$ with dimension 1, which correspond to irreducible curves. $Z_2(A)$ is free of rank 1 on the class of $[A]$, which corresponds to the 0 ideal.

Let $M$ be an $A$-module; we wish to define a cycle associated to $M$. While it appears natural to take the set of associated or minimal primes, it turns out to be more useful to make a slightly different choice. Let the dimension of $M$ be at most $i$. Then if $\mathfrak{p}$ is any prime ideal such that $\dim(A/\mathfrak{p}) = i$, $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module of finite length (possible zero). Denote the length of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module by length$(M_\mathfrak{p})$. We let the cycle of dimension $i$ associated to $M$, denoted $[M]_i$, be the sum

$$\sum_{\dim(A/\mathfrak{p})=i} \text{length}(M_\mathfrak{p})[A/\mathfrak{p}].$$

In Chapter 12 we will define a more complicated cycle associated with a module that has better naturality properties. In many important cases it will agree with the one defined here, but not in general.

We note that we could have summed only over those $\mathfrak{p}$ of dimension $i$ in the support of $M$, since the coefficients of $[A/\mathfrak{p}]$ for $\mathfrak{p}$ not in the support of $M$ are zero. Also note that if $M = A/\mathfrak{p}$ and $i$ is the dimension of $A/\mathfrak{p}$, then $[A/\mathfrak{p}]_i$ is the same as the generator $[A/\mathfrak{p}]$ defined above. If the dimension of $M$ is less than $i$, then $[M]_i = 0$.

We next define the Chow group, which is obtained by dividing the group $Z_*(\text{Spec}(A))$ of cycles by a certain equivalence relation, which we now describe. Let $\mathfrak{p}$ be a prime ideal of dimension $i + 1$, and let $x$ be an element of $A$ that is not in $\mathfrak{p}$. Then, since the support of $(A/\mathfrak{p})/x(A/\mathfrak{p})$ does not contain the unique minimal prime ideal in Spec$(A/\mathfrak{p})$, the dimension of $(A/\mathfrak{p})/x(A/\mathfrak{p})$ is at most $i$. We denote the cycle $[(A/\mathfrak{p})/x(A/\mathfrak{p})]_i$ by div$(\mathfrak{p}, x)$.

**Definition 1.2.1** Rational equivalence *is the equivalence relation on* $Z_i(A)$ *generated by setting* div$(\mathfrak{p}, x) = 0$ *for all prime ideals* $\mathfrak{p}$ *of dimension* $i + 1$ *and all elements* $x \notin \mathfrak{p}$. *In other words, two cycles are rationally equivalent if their difference lies in the subgroup generated by the cycles of the form* div$(\mathfrak{p}, x)$.

*The* Chow group *of A is the direct sum of the groups* $A_i(A)$, *where* $A_i(A)$ *is the group of cycles* $Z_i(A)$ *modulo rational equivalence.*

The Chow group is denoted $A_*(A)$ or $A_*(\text{Spec}(A))$. Like the group of cycles, it comes with a natural grading by dimension.

As an example, we describe the Chow group of $k[X, Y]$, where $k$ is an algebraically closed field. We have that $A_2(k[X, Y]) \cong \mathbb{Z}$ since there are no prime ideals of dimension 3 and thus no relations in dimension 2. Every prime ideal of dimension 1 is principal (since $k[X, Y]$ is a unique factorization domain), and hence every generator $[A/\mathfrak{p}]$ of $Z_1(A)$ is of the form $[A/fA] =$ div$(0, f)$ where $f$ generates $\mathfrak{p}$, so $A_1(A) = 0$. Every maximal ideal $\mathfrak{m}$ is generated by two elements $x - a, y - b$ with $a$ and $b$ in $k$. Then $[A/\mathfrak{m}] =$ div$((x - a), y - b)$, so we also have $A_0(A) = 0$.

While the definition often makes it easy to check that an element of the Chow group is zero, it is usually more difficult to show that elements are not zero. One simple example is given by minimal prime ideals; since the cycle $\mathrm{div}(\mathfrak{q}, x)$ has nonzero coefficients only for nonminimal prime ideals, for any minimal prime ideal $\mathfrak{p}$, the class $[A/\mathfrak{p}]$ is not zero, and is in fact not torsion, in $A_*(A)$. We next give an extended example where an alternative construction is available, and which provides less trivial examples in which cycles can be shown not to vanish in $A_*(A)$.

Let $A$ be an integrally closed domain of dimension $d$. We show that $A_{d-1}(A)$ can be expressed in terms of the divisor class group of $A$.

Let $K$ be the quotient field of $A$. A nonzero finitely generated $A$-submodule of $K$ is called a *fractional ideal*. A nonzero ideal of $A$ is a special case of a fractional ideal, and if $\mathfrak{a}$ is a fractional ideal, since $\mathfrak{a}$ is finitely generated, there exists a nonzero element $x \in A$ such that $x\mathfrak{a}$ is an ideal in $A$. We call a fractional ideal $\mathfrak{a}$ *divisorial* if $K/\mathfrak{a}$ has no associated prime ideals of dimension less than $d-1$. We note that a prime ideal has dimension $d-1$ if and only if its height is 1.

If $A$ is a Noetherian integrally closed domain and $\mathfrak{p}$ is a prime ideal of $A$ of dimension $d-1$, then the localization $A_\mathfrak{p}$ is a discrete valuation ring (see Atiyah and Macdonald [1, Chapter 9]). Thus, for any fractional ideal $\mathfrak{a}$ of $A$, the localization $\mathfrak{a}_\mathfrak{p}$ is a (possibly negative) power of the maximal ideal of $A_\mathfrak{p}$; we denote the power $v_\mathfrak{p}(\mathfrak{a})$. If $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals, then the product $\mathfrak{a}\mathfrak{b}$ is also a fractional ideal, and we have

$$v_\mathfrak{p}(\mathfrak{a}\mathfrak{b}) = v_\mathfrak{p}(\mathfrak{a}) + v_\mathfrak{p}(\mathfrak{b})$$

for all $\mathfrak{p}$. If $x$ is an element of $K$, we define $v_\mathfrak{p}(x)$ to be $v_\mathfrak{p}(Ax)$. Then $v_\mathfrak{p}(x) = 0$ for all but finitely many $\mathfrak{p}$, and $x \in A$ if and only if $v_\mathfrak{p}(x) \geq 0$ for all $\mathfrak{p}$.

We next show that the ring $A$ itself, and in fact any principal fractional ideal, is divisorial. The proof uses the fact that a proper principal ideal cannot have height greater than one; this result will be proven in Chapter 2 (Theorem 2.3.8).

**Theorem 1.2.1** *Let $A$ be a Noetherian integrally closed domain. Then every principal fractional ideal is divisorial.*

*Proof* Let $x/y$ be an element of the quotient field of $A$, and suppose that $K/(x/y)A$ has an associated prime ideal $\mathfrak{p}$ of height greater than one. Localizing at $\mathfrak{p}$, we may assume that $A$ is local and that $\mathfrak{p}$ is its maximal ideal. Let $w/z$ be an element of $K$ whose annihilator modulo $(x/y)A$ is $\mathfrak{p}$. We then have

$$\mathfrak{p}(w/z) \subseteq (x/y)A \quad \text{but} \quad w/z \notin (x/y)A.$$

Multiplying these equations by $y/x$, we see that they are equivalent to

$$\mathfrak{p}(yw/zx) \subseteq A \quad \text{but} \quad yw/xz \notin A.$$

Hence it suffices to show that $A$ itself is divisorial. We thus assume that $x/y = 1$.

Since $\mathfrak{p}$ is the maximal ideal of $A$ and $\mathfrak{p}(w/z) \subseteq A$, we either have $\mathfrak{p}(w/z) = A$ or $\mathfrak{p}(w/z) \subseteq \mathfrak{p}$. If $\mathfrak{p}(w/z) \subseteq \mathfrak{p}$, then $w/z$ is integral over $A$ ([1, Proposition 2.4]), so $w/z \in A$ since $A$ is integrally closed. Since we are assuming that $w/z \notin A$, we thus cannot have $\mathfrak{p}(w/z) \subseteq \mathfrak{p}$. Hence it remains to show that the alternative, that $\mathfrak{p}(w/z) = A$, is also impossible.

If $\mathfrak{p}(w/z) = A$, then there exists an $a \in \mathfrak{p}$ such that $a(w/z) = 1$. If $b$ is any element of $\mathfrak{p}$, then $b(w/z) = c$ for some $c \in A$, so $b(w/z) = ca(w/z)$ and thus $b = ca$. But this argument shows that $\mathfrak{p}$ is a principal ideal generated by $a$, which contradicts the assumption that $\mathfrak{p}$ has height at least two. Thus, $A$ is divisorial, and hence every principal fractional ideal is divisorial. □

We next define a map $\phi$ from the set of divisorial ideals to the group of cycles $Z_{d-1}(A)$ by letting

$$\phi(\mathfrak{a}) = \sum v_{\mathfrak{p}}(\mathfrak{a})[A/\mathfrak{p}],$$

where the sum runs over all prime ideals of height one. Since $\mathfrak{a}_{\mathfrak{p}} = A_{\mathfrak{p}}$ for all but finitely many $\mathfrak{p}$ of height one, all but finitely many $v_{\mathfrak{p}}$ are zero and the sum is finite.

We next show that $\phi$ is a bijection. If $\mathfrak{a}$ and $\mathfrak{b}$ are divisorial ideals, then every associated prime of $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{b}$ or $\mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}$ has height 1, so $\mathfrak{a} = \mathfrak{b}$ if and only if $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of height one; thus $\phi$ is injective. We claim that $\phi$ is also surjective. Suppose first that $\eta = \sum n_{\mathfrak{p}}[A/\mathfrak{p}]$ is an element of $Z_{d-1}(A)$ such that every $n_{\mathfrak{p}}$ is nonnegative. Let $f$ be the natural map from $A$ to the product of $A_{\mathfrak{p}}$ over those $\mathfrak{p}$ of height one for which $n_{\mathfrak{p}} \neq 0$, and let $\mathfrak{a}$ be the inverse image under $f$ of the product of $\mathfrak{p}_{\mathfrak{p}}^{n_{\mathfrak{p}}}$. Then $\mathfrak{a}$ is a divisorial ideal and $\phi(\mathfrak{a}) = \eta$.

If some of the coefficients $n_{\mathfrak{p}}$ are negative, we first find an element $x \in A$ such that $v_{\mathfrak{p}}(x) \geq -n_{\mathfrak{p}}$ for all $\mathfrak{p}$; for example, we can choose $x$ to be any nonzero element in the product of $\mathfrak{p}^{-n_{\mathfrak{p}}}$ over those $\mathfrak{p}$ for which $n_{\mathfrak{p}}$ is negative. Then $v_{\mathfrak{p}}(x) + n_{\mathfrak{p}} \geq 0$ for all $\mathfrak{p}$, so there exists an ideal $\mathfrak{a}$ of $A$ such that $v_{\mathfrak{p}}(\mathfrak{a}) = v_{\mathfrak{p}}(x) + n_{\mathfrak{p}} \geq 0$ for all $\mathfrak{p}$ of height one. Thus, $\phi(x^{-1}\mathfrak{a}) = \sum n_{\mathfrak{p}}[A/\mathfrak{p}]$, so $\phi$ is surjective.

Since the map $\phi$ is a bijection, we can give the set of divisorial ideals the structure of an Abelian group by pulling back the structure on $Z_{d-1}(A)$. Since $v_{\mathfrak{p}}(\mathfrak{a}\mathfrak{b}) = v_{\mathfrak{p}}(\mathfrak{a}) + v_{\mathfrak{p}}(\mathfrak{b})$ for fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$, this structure is essentially the same as multiplication of ideals; however, there is no reason why