Choosing the most efficient statistical test is one of the basic problems of statistics. Asymptotic efficiency is an indispensable technique for comparing and ordering statistical tests in large samples. It is especially useful in nonparametric statistics where there exist numerous heuristic tests such as the Kolmogorov–Smirnov, Cramér–von Mises, and linear rank tests.

This monograph discusses the analysis and calculation of the asymptotic efficiencies of nonparametric tests. Powerful methods based on Sanov's theorem together with the techniques of limit theorems, variational calculus, and nonlinear analysis are developed to evaluate explicitly the large-deviation probabilities of test statistics. This makes it possible to find the Bahadur, Hodges–Lehmann, and Chernoff efficiencies for the majority of nonparametric tests for goodness-of-fit, homogeneity, symmetry, and independence hypotheses.

Of particular interest is the description of domains of the Bahadur local optimality and related characterization problems, based on recent research by the author. The general theory is applied to a classical problem of statistical radio physics: signal detection in noise of unknown level. Other results previously published only in Russian journals are also published here for the first time in English.

Researchers, professionals, and students in statistics will find this unified treatment of the subject invaluable.

ASYMPTOTIC EFFICIENCY OF NONPARAMETRIC TESTS

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YAKOV NIKITIN
St. Petersburg State University

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Introduction

Choosing the most efficient statistical test of several ones that are at the disposal of the statistician is regarded as one of the basic problems of statistics. According to the classical Neyman–Pearson theory the uniformly most powerful tests are considered the best. However, it is well known that they exist merely for a narrow class of statistical models which do not fully cover the diversity of problems arising in theory and practice. One can still say that within the framework of parametric statistics this problem is not at all crucial. The point is that quite formal methods of constructing tests have been developed, for example, Bayes or likelihood ratio tests. They possess a number of remarkable properties and usually turn out to be asymptotically optimal in the sense of one or another definition of this concept.

The situation is quite different under the nonparametric approach. There exist numerous statistical tests proposed as a rule for heuristic reasons. The Kolmogorov–Smirnov and omega-square tests can serve as classical examples for goodness-of-fit testing. In other cases nonparametric procedures arise as simple substitutes of computationally complicated parametric procedures. The Wilcoxon rank test has been proposed in exactly this way. One more reason for using nonparametric tests is concerned with unreliable information on the distribution of observations in cases when it is reasonable to use, instead of the highly suitable parametric test, a nonparametric one, which is possibly less efficient but more robust with respect to changes of this distribution.

Because of these and some other reasons there has resulted an extraordinary diversity of nonparametric procedures. In the early sixties the well-known bibliography of nonparametric statistics by Savage (1962) contained already about 3,000 entries. The idea of this diversity was also presented in the three-volume Walsh handbook (1962, 1965, 1968) and the monograph by Hollander and Wolfe (1973) as well as in the reference *Handbook of Statistics* (Krishnaiah and Sen 1984).

Therefore the problem of comparing nonparametric tests on the basis of some quantitative characteristic that will make it possible to order these tests and recommend the proper test one should use in the given problem becomes extremely important. The asymptotic efficiency is just the most known and useful characteristic of such kind.

Kendall and Stuart (1967) pointed out that the notion of the asymptotic efficiency of tests is more complicated than the asymptotic efficiency of estimates. Various approaches to this notion were identified only in the late forties and early fifties, hence, 20 to 25 years later than for the estimation theory. We proceed now to their description.

Let $\{T_n\}$ and $\{V_n\}$ be two sequences of statistics based on n observations and assigned for testing the null or basic hypothesis H against the alternative A. We assume that the alternative is characterized by parameter θ and for $\theta = \theta_0$ turns into H. Denote by $N_T(\alpha, \beta, \theta)$ the sample size necessary for the sequence $\{T_n\}$ in order to attain the power β under the level α and the alternative value of parameter θ. The number $N_V(\alpha, \beta, \theta)$ is defined in the same way. The relative efficiency of the sequences $\{T_n\}$ with respect to the sequence $\{V_n\}$ is specified as the quantity

$$e_{T,V}(\alpha, \beta, \theta) = N_V(\alpha, \beta, \theta) / N_T(\alpha, \beta, \theta)\,,$$

so it is the reciprocal ratio of sample sizes N_T and N_V.

The merits of the relative efficiency as a means for comparing the tests are beyond any doubt. Unfortunately it is extremely difficult to explicitly compute $N_T(\alpha, \beta, \theta)$ even for the simplest sequences of statistics $\{T_n\}$. At present it is recognized that it is possible to avoid this difficulty by calculating the limiting values $e_{T,V}(\alpha, \beta, \theta)$ as $\theta \to \theta_0$, as $\beta \to 1$, and as $\alpha \to 0$, keeping the two other parameters fixed. These limiting values are called respectively the Pitman, Hodges–Lehmann, and Bahadur asymptotic relative efficiency (ARE).

Only close alternatives, high powers, and small levels are of the most interest from the practical point of view. This assures one that the knowledge of these ARE types will facilitate comparing concurrent tests, thus producing well-founded application recommendations.

Coupled with the three "basic" approaches to the ARE calculation just described, intermediate approaches are also possible if the transition to the limit occurs simultaneously for two parameters in a controlled way.

Thus emerged the Chernoff ARE introduced by Chernoff (1952) and the intermediate, or Kallenberg, ARE introduced by Kallenberg (1983a).

The history of nonparametric statistics contains plenty of good examples when the calculation of the AREs of some sequences of tests has resulted in the reappraisal of their merits. For instance, in the forties and fifties a very popular homogeneity test of two samples was the run test of Wald and Wolfowitz (see, e.g., Wald and Wolfowitz (1940) or Noether (1950)). But Mood (1954) proved that its Pitman ARE with respect to parametric tests is equal to 0. Later Bahadur (1960b) stated that the same is also true in the case of the Bahadur efficiency. In subsequent years the run test was gradually discarded from the statistical literature and replaced by more efficient rank tests. The same has occurred in the case of goodness-of-fit tests based on spacings according to Chibisov (1961).

The Pitman efficiency introduced in the late forties by Pitman (1949) is the most well-known and carefully studied type of ARE (see also Pitman (1979)). The principal condition ensuring the possibility of its computation is the asymptotic normality of a given sequence of statistics under the null hypothesis and the alternative. In this case the Pitman ARE is a number not depending on α and β. It has been computed in numerous papers. The results obtained there are partly referred to in corresponding sections of the monographs by Kendall and Stuart (1967), Lehmann (1959, 1975), and Hollander and Wolfe (1973). At present one can say that the Pitman ARE is computed for the majority of pairs of nonparametric statistics having asymptotically normal distributions.

The situation becomes much more complicated when a given sequence of statistics has different limiting distributions under the null hypothesis and the alternative such that at least one of them is nonnormal. These are the cases of the Kolmogorov–Smirnov and ω^2 statistics, when the Pitman ARE can depend on α and β and it is difficult to determine its value. The most promising results have been obtained by Wieand (1976) who modified the definition of the ARE and found its limiting (as $\alpha \to 0$) value for most frequently used statistics. For that reason we do not pay much attention to the Pitman ARE in the present book.

The present monograph deals with the calculation and analysis in the Bahadur, Hodges–Lehmann, and Chernoff senses of the ARE of Kolmogorov–Smirnov and Cramér–von Mises nonparametric tests and their variants as well as linear rank tests for testing goodness-of-fit, homogeneity, symmetry, and independence. The conditions of the local asymptotic optimality of these tests are also considered here.

The calculation of the Bahadur, Hodges–Lehmann, and Chernoff efficiencies, unlike the Pitman efficiency, is based on rough asymptotics of large-deviation probabilities of a given sequence of statistics under the null hypothesis and the alternative. The approaches to calculating the ARE dealing with the analysis of large deviations have become widespread. The following citation from Dacunha-Castelle (1979) is significant: "The merit of large deviations consists in that they give the solution for the problems of classification of tests and the search of optimal solutions when ... classical methods do not give any answer. The application of such a technique seems methodologically quite justified." Convincing arguments in favor of the use of large deviations by asymptotic comparison of tests are given also by Bahadur (1960), Neyman (1980), Serfling (1980), Borovkov and Mogulskii (1992), and others.

The theory of large deviations has reached by now a state of considerable development that has been stimulated in many respects by the demands of statistics. Two lines of research in the study of large deviations of nonparametric statistics are evident among the multitude of publications in this area (see Chapter 1 for more a detailed exposition).

The first considers various generalizations and continuations of results by Cramér (1938) and Chernoff (1952) who analyzed sums of independent identically distributed random variables. It turned out that the achievements in this direction made it possible to find large-deviation asymptotics of the Kolmogorov–Smirnov statistics and their variants under the null hypothesis. See Abrahamson (1967), Bahadur (1971), Groeneboom and Shorack (1981), Nikitin (1987b), and Podkorytova (1990).

The second line is based on the fundamental paper by Sanov (1957) exploring large deviations of empirical distribution functions. The Sanov results were generalized and developed later by Hoadley (1967); Groeneboom, Oosterhoff, and Ruymgaart (1979); Fu (1985); and others. It is important to mention the paper of Borovkov (1967) where the author noted that the Sanov-type theorems "permit estimating the probabilities of large deviations, for instance, of ω^2 statistics and comparing the power of corresponding tests under various alternatives."

The rough asymptotics of large-deviation probabilities for the classical Cramér–von Mises–Smirnov goodness-of-fit statistic ω_n^2 under the null hypothesis have been determined by Mogulskii (1977) by applying the results of Borovkov (1967). This statistic is as follows:

$$\omega_n^2 = \int_0^1 \left(F_n(t) - t \right)^2 dt, \qquad (1)$$

where F_n is the empirical distribution function based on a sample from the uniform distribution on [0, 1]. It is well known that the Sanov theorem and its generalizations reduce the problem of large deviations to the minimization problem of the Kullback–Leibler information on the corresponding set of distribution functions. The latter belongs to the class of variational problems on conditional extremum. The extremal may be found from the Euler–Lagrange equation which is a nonlinear differential equation of the second order having the form

$$x'' - \lambda x - \lambda \varepsilon x x' = 0 \tag{2}$$

and should be considered jointly with the conditions:

$$x(0) = x(1) = 0, \qquad \int_0^1 x^2(t)\, dt = 1. \tag{3}$$

In equation (2) λ is an indeterminate Lagrange multiplier and ε is a small numerical parameter.

Mogulskii (1977) proved by using the specific form of equation (2) that the solution of problem (2)–(3) may be found in the class of functions

$$x(t) = \sum_{n=0}^{\infty} x_n(t)\, \varepsilon^{n/2}, \tag{4}$$

where

$$x_n(t) = \sum_{k=1}^{\infty} a_k^{(n)} \sin k\pi t,$$

and the series in (4) converges absolutely and uniformly for sufficiently small $\varepsilon > 0$. This fact enables one to prove that

$$\lim_{n\to\infty} n^{-1} \ln P(\omega_n^2 \geq \varepsilon) = -\tfrac{1}{2}\pi^2 \varepsilon + \sum_{k=3}^{\infty} c_k\, \varepsilon^{k/2}, \tag{5}$$

where the series on the right-hand side converges for sufficiently small positive ε.

However, the arguments of Mogulskii (1977) cannot be adapted even for the case of weighted statistics, namely,

$$\omega_{n,q}^2 = \int_0^1 \big(F_n(t) - t\big)^2 q(t)\, dt, \tag{6}$$

where q is a summable nonnegative weight function on [0, 1]. This is concerned, to a considerable extent, with the complicated integral statistics for testing homogeneity, symmetry, and independence when the Euler–Lagrange equations are much more involved.

Therefore we propose to use a more general method to analyze these equations. The essence of the matter is to consider equation (2) or its more complicated counterparts together with the boundary conditions as an implicit analytic operator acting in a suitably chosen Banach space. It is quite natural to expect the solutions of such equations, as (4), to be analytic in a small parameter ε or in a certain power of it. The important obstacle in proving this is the fact that the linear part (the Fréchet derivative for $\varepsilon = 0$) of all operators under consideration is noninvertible. Therefore, we use the theory of branching of the solutions of nonlinear equations (see Vainberg and Trenogin (1974)), due to Lyapunov and Schmidt, to rule out the existence of bifurcations. The joint analysis of the so-called branching equation with the normalization condition enables one to construct the solutions of the Euler–Lagrange equations in the form of power series in a small parameter. Moreover, their convergence in a neighborhood of zero follows automatically from the general theorems proved by Vainberg and Trenogin (1974).

After the substitution of these solutions into the minimized functional one obtains the expected asymptotics. Further we compute the local Bahadur efficiency of various integral statistics by using the indicated method as a basis.

The computation of the Hodges–Lehmann and Chernoff AREs meets with considerable difficulties because of the necessity to study large deviations of nonparametric statistics under the alternative when they are no longer "distribution-free." However, it has become possible to overcome these difficulties by modifying the method described earlier. This enables one to study the behavior of the Hodges–Lehmann and Chernoff indices of a large class of linear rank statistics and to prove their local coincidence with the Bahadur exact slopes. Taking into account that the local Bahadur efficiency is coincident with respect to these statistics with the Pitman and intermediate efficiencies (see Kremer (1979a,b) and Kallenberg (1983a)), one discovers an interesting phenomenon, namely, that the local asymptotic ordering of a large class of linear rank tests does not depend on the ARE type.

The situation turns out to be entirely different for the Kolmogorov–Smirnov and omega-square statistics. Combining general results on large deviations of empirical measures under the alternative with purely statistical reasons, we prove that the two-sided tests of Kolmogorov–Smirnov and ω^2 type attain the maximum possible Hodges–Lehmann ARE in all problems under consideration and thus are somewhat unexpectedly asymptotically optimal. Even the one-sided Smirnov test may possess this property under appropriate conditions. Local Chernoff indices of these statistics have complicated expressions and their values are not, as a rule, asymptotically optimal.

It is well known that the Bahadur exact slopes and Hodges–Lehmann indices of any sequence of statistics are bounded from above by a positive quantity whose value can be specified in terms of the Kullback–Leibler information. This was discovered by Stein for simple hypotheses; later, various generalizations were made by Rao (1962), Raghavachari (1970), Bahadur and Raghavachari (1972), Nikitin (1986c), and Kourouklis (1988). In other words a statistical test cannot be altogether "too good": Just like in regular experiments the estimates of a parameter cannot be "too precise" (the Cramér–Rao inequality). It has been proved that under some natural auxiliary conditions the likelihood ratio tests are asymptotically optimal in the Bahadur sense (see Bahadur (1965) and Rublik (1989)) as well as in the Hodges–Lehmann sense (see Brown (1971)) and Chernoff sense (see Kallenberg (1982)). It remains to be seen if this is possible for nonparametric tests.

We investigate this problem with regard to the Kolmogorov–Smirnov-type tests and ω^2-type tests as well as for linear rank tests. We find that these tests may possess the property of Bahadur *local* optimality. Relying on expressions for local exact slopes one succeeds in obtaining the description of the structure of those families of distributions for which a given nonparametric test is locally asymptotically optimal in the Bahadur sense (the domain of the local asymptotic optimality will be given in our terminology). A key part in this description is played by the "leading" sets of functions which are specific for each test and indicate, roughly speaking, those "directions" in the set of alternatives where a given test is most sensitive.

If some auxiliary information is available on the structure of the initial family of distributions (i.e., location, scale, Lehmann families, etc), the conditions of local asymptotic optimality lead to original characterization theorems. For instance, for the location families the Kolmogorov goodness-of-fit test is locally asymptotically optimal in the Bahadur sense only for the Laplace distribution. Likewise, the ω^2 test is Bahadur locally optimal only for the "hyperbolic cosine" distribution and the Wilcoxon two-sample rank test possesses this property only for the logistic one. These results are partially preserved for other types of asymptotic efficiency.

The nonparametric tests under consideration are widely used in solving numerous problems of natural sciences and engineering. Their asymptotic efficiency facilitates the choice of the most efficient test and therefore meets the requirements of statistical practice. The asymptotic comparison of tests in such a high-priority problem of statistical radio physics as the problem of detection of signals in Gaussian noise of unknown level in order to apply them in concrete conditions is considered at the end of Chapter 3.

The monograph consists of the Introduction and six chapters divided into sections. We adopt a triple numbering convention, with formulas, theorems, and lemmas being independent in each section. Chapter 1 has an auxiliary character. There the mathematical results used subsequently for the calculation and analysis of the asymptotic efficiency are put together. The next four chapters deal with large deviations and asymptotic efficiency of nonparametric tests. We consider goodness-of-fit, homogeneity, symmetry, and independence tests in Chapters 2, 3, 4, and 5, respectively. The final chapter contains the description of the local asymptotic optimality domains of nonparametric statistics and corresponding characterization results.

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