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Asymptotic Efficiency of Statistical Tests and Mathematical Means for Its Computation

1.1 General Approach to Computation of Asymptotic Efficiency

Let $(\mathfrak{X}, \mathfrak{A})$ be a sample space corresponding to the observation X. It is assumed that the distribution P_θ of this observation is determined by parameter θ taking on values in a parametric set Θ. Let $s = \{X_1, X_2, \ldots\}$ be a sequence of independent identically distributed random variables with values in $\mathfrak{X}$ and having the distribution P_θ on $\mathfrak{A}$. For any positive integer n put $\mathbf{X}^{(n)} := (X_1, X_2, \ldots, X_n)$ and denote by $(\mathfrak{X}^{(n)}, \mathfrak{A}^{(n)})$ the corresponding sample space and by $P_\theta^{(n)}$ the distribution of $\mathbf{X}^{(n)}$ on $\mathfrak{A}^{(n)}$. In the sequel $P_\theta^{(n)}$, $1 \le n \le \infty$, will be usually abbreviated to P_θ.

Consider the problem of testing the hypothesis

$$H\colon\ \theta \in \Theta_0 \subset \Theta$$

against the alternative

$$A\colon\ \theta \in \Theta_1 = \Theta \setminus \Theta_0$$

on the basis of observations $X_1, X_2, \ldots, X_n$. For this purpose we use a sequence of statistics $\{T_n\}$, $T_n(s) := T_n(X_1, X_2, \ldots, X_n)$, assuming (without essential loss of generality) large values of T_n to be significant. Thus the critical or rejection region of H is given by

$$\{\, s\colon\ T_n(s) \ge c \,\},$$

where c is some real number.

The *power function* of this test is the quantity $P_\theta(T_n \ge c)$ considered as a function of θ and its *size* is equal to

$$\sup\{\, P_\theta\,(T_n \ge c)\colon\ \theta \in \Theta_0 \,\}.$$

Now define for any $\beta \in (0,\ 1)$ and $\theta \in \Theta_1$ a real sequence $c_n := c_n(\beta, \theta)$ with the aid of double inequality

$$P_\theta\left(T_n > c_n\right) \leq \beta \leq P_\theta\left(T_n \geq c_n\right). \tag{1.1.1}$$

Then

$$\alpha_n(\beta, \theta) := \sup\left\{\, P_{\theta'}(T_n \geq c_n) \colon\ \theta' \in \Theta_0 \,\right\}$$

is the minimal size of the test based on $\{T_n\}$ for which the power at the point θ is not less than β. Let us define for any given level of significance α, $0 < \alpha < \beta$, the positive integer

$$N_T(\alpha, \beta, \theta) := \min\left\{n\colon\ \ \alpha_m(\beta, \theta) \leq \alpha \quad \text{for all } m \geq n\right\}.$$

It is clear that $N_T(\alpha, \beta, \theta)$ is the minimal sample size necessary for the test at a level α, based on $\{T_n\}$, to have the power not less than β at the point θ.

Suppose that for testing H against A we have two sequences of test statistics $\{T_n\}$ and $\{V_n\}$. Define by $e_{V,T}(\alpha, \beta, \theta)$ the *relative efficiency* of the sequence $\{V_n\}$ with respect to $\{T_n\}$ in the following way:

$$e_{V,T}(\alpha, \beta, \theta) := N_T(\alpha, \beta, \theta) / N_V(\alpha, \beta, \theta). \tag{1.1.2}$$

A value $e_{V,T}(\alpha, \beta, \theta)$ larger than 1 indicates that for given α, β, and θ one should prefer the sequence $\{V_n\}$ to $\{T_n\}$ because the first sequence requires less observations for reaching the power β for the level α and the alternative value θ.

As has been already noted in the Introduction, relative efficiency (1.1.2) has indisputable merits. Unfortunately it also has two substantial drawbacks. One consists in that the value of $e_{V,T}(\alpha, \beta, \theta)$ depends on three arguments (and two sequences of statistics), the other is connected with the fact that it is extremely difficult or simply impossible to calculate this value. It is possible at present to overcome these difficulties by calculating the limiting values of $e_{V,T}(\alpha, \beta, \theta)$ as $\alpha \to 0$, as $\beta \to 1$, and as $\theta \to \theta_0 \in \partial\Theta_0$ (in a certain topology on Θ) keeping fixed the values of two remaining parameters. As a result one obtains three fundamental types of the asymptotic relative efficiency (ARE).

If for $\beta \in (0,\ 1)$ and $\theta \in \Theta_1$ there exists the limit

$$e^B_{V,T}(\beta, \theta) := \lim_{\alpha \downarrow 0} e_{V,T}(\alpha, \beta, \theta), \tag{1.1.3}$$

it is called the *Bahadur ARE of the sequence* $\{V_n\}$ *with respect to* $\{T_n\}$.

If for $\alpha \in (0, 1)$ and $\theta \in \Theta_1$ there exists the limit

$$e^{HL}_{V,T}(\alpha, \theta) := \lim_{\beta \uparrow 1} e_{V,T}(\alpha, \beta, \theta) , \qquad (1.1.4)$$

it is called the *Hodges–Lehmann ARE of the sequence* $\{V_n\}$ *with respect to* $\{T_n\}$.

If for $0 < \alpha < \beta < 1$ and $\theta \to \theta_0 \in \partial\Theta_0$ (in a certain topology on Θ) there exists the limit

$$e^{P}_{V,T}(\alpha, \beta, \theta_0) := \lim_{\theta \to \theta_0} e_{V,T}(\alpha, \beta, \theta) , \qquad (1.1.5)$$

it is called the *Pitman ARE of the sequence* $\{V_n\}$ *with respect to* $\{T_n\}$.

It is also difficult to calculate these three types of the ARE, but it is still much easier than relative efficiency (1.1.2). Moreover, the Bahadur ARE usually does not depend on β, the Hodges–Lehmann ARE does not depend on α, and the Pitman ARE in most cases depends neither on α nor on β and turns out to be a constant. We emphasize once again that from the practical point of view the very cases of small levels, high powers, and close alternatives are the most important. That is why one may suppose that the knowledge of three types of the ARE such as (1.1.3)–(1.1.5) will in a sense help to put in order principal tests used in a concrete problem and will permit the well-founded recommendations for their applications in practice.

Note that there exist the intermediate approaches to measuring the ARE not coinciding with the approaches of Bahadur, Hodges and Lehmann, and Pitman. The typical examples are the Chernoff ARE (see Chernoff (1952), Kallenberg (1982), and Ronzhin (1985)) when for a fixed θ the other parameters α and β tend to 0, and the case of the intermediate or Kallenberg ARE (see Kallenberg (1983a)), when β is fixed, but θ and α tend to θ_0 and 0 at a controlled rate.

The different definitions of the ARE belong also to Rubin and Sethuraman (1965b) and to Borovkov and Mogulskii (1992). The first exploits the notion of the Bayes risk whereas the second deals with a certain modification of the number $N_T(\alpha, \beta, \theta)$, being more symmetric with respect to the errors of the first and second kinds. Their values for more or less complicated nonparametric statistics are unknown.

It would be most interesting to learn if the values of AREs calculated under different approaches are close to the values of the relative efficiency for finite samples and reasonable values of α, β, and θ arising in practical problems. The first such comparison has been realized by Groeneboom and Oosterhoff (1981) with the aid of statistical modeling. The results of Groeneboom and Oosterhoff (1981) are connected with some simplest examples only and for the present do not give any reasons for definite conclusions.

It may happen that the value of the ARE is equal to 1 and this circumstance prevents the asymptotic comparison of tests. In that case one may recommend following Hodges and Lehmann (1970) and using more sensitive means for comparing different tests, namely, the *deficiency*

$$\operatorname{def}(V,T;\alpha,\beta,\theta) := N_V(\alpha,\beta,\theta) - N_T(\alpha,\beta,\theta)\,, \quad \theta\in\Theta_1\,.$$

Asymptotic approximations to the deficiency as $\alpha\to 0$ and $\theta\to\theta_0\in\partial\Theta_0$ have been considered, mainly for parametric statistics, by Albers (1974), Chandra and Ghosh (1978), Groeneboom and Oosterhoff (1981), and Kallenberg (1981, 1982) as well as by Borovkov and Mogulskii (1992). Not much is known in the nonparametric case and we will not use the notion of the deficiency in the sequel.

1.2 Bahadur Asymptotic Relative Efficiency

The Bahadur approach to measuring the ARE is opposite to the classical approach of Neyman and Pearson and prescribes one to fix the power of tests and to compare the rate of decreasing of their sizes for the increasing number of observations. This point of view, expressed for the first time by Cochran (1952), has been deeply and systematically developed by Bahadur (1960a,b, 1967, 1971). Other expositions of the Bahadur theory with reviews of publications in this area may be found in Savage (1969), Groeneboom and Oosterhoff (1977), and Serfling (1980).

Denote for any θ, t and any sequence of statistics $\{T_n\}$

$$F_n(t;\theta) := P_\theta\left(s\colon\ T_n(s) < t\right), \qquad G_n(t) := \inf\left\{F_n(t;\theta)\colon\ \theta\in\Theta_0\right\}.$$

The quantity

$$L_n(s) := 1 - G_n(T_n(s))$$

is called the *attained level* or the *P-value*. This is a random variable representing the degree to which the test statistic T_n rejects H.

For $\theta\in\Theta_0$ the P-value is distributed approximately uniformly on $[0,\ 1]$. Anyway the following inequality is valid:

$$P_\theta\left(L_n\le u\right)\le u \qquad \text{for any } u\in[0,\ 1]\,. \tag{1.2.1}$$

In the case of continuous distribution function $F_n(t;\theta_0)$ this inequality follows directly from the definition of L_n; the general case is based on a suitable approximation (see Bahadur (1971), Theorem 7.4). Therefore working with finite samples one compares the P-value with the preassigned level α and rejects the basic hypothesis if $L_n < \alpha$. Substantial discussion and the interpretation of P-values were given by Gibbons and Pratt (1975), Serfling (1980), Lambert and Hall (1982), and Chandra (1989).

The asymptotic behavior of L_n under the alternative ($\theta \in \Theta_1$) is of considerable interest for comparing sequences of test statistics. It is usually the case that for $\theta \in \Theta_1$ the following convergence in P_θ-probability takes place:

$$\lim_{n\to\infty} n^{-1} \ln L_n = -\tfrac{1}{2}\, c_T(\theta), \tag{1.2.2}$$

where $c_T(\theta)$ is a nonrandom positive function of parameter θ on Θ_1, that is called the *Bahadur exact slope of the sequence* $\{T_n\}$. The factor $\frac{1}{2}$ is present in (1.2.2) due to historical reasons. Some authors, for example, Groeneboom and Oosterhoff (1977) as well as Chandra and Ghosh (1978), have called $c_T(\theta)$ the *weak slope*, in contrast to the *strong slope* when the convergence in (1.2.2) takes place with P_θ-probability 1. In the sequel we will use the term slope mainly for weak slopes as it is sufficient to consider the convergence in probability in most statistical problems.

One may rewrite (1.2.2) in terms of sample sizes $N_T(\alpha,\beta,\theta)$.

Theorem 1.2.1 (Bahadur 1967; Groeneboom and Oosterhoff 1977) *If (1.2.2) is valid for a sequence of statistics* $\{T_n\}$ *with* $c_T(\theta) > 0$, *then*

$$N_T(\alpha,\beta,\theta) \sim \frac{2\ \ln 1/\alpha}{c_T(\theta)} \qquad \text{as } \alpha \to 0\,. \tag{1.2.3}$$

Proof Let us prove at first that for any $\beta \in (0,\, 1)$ we have

$$\lim_{n\to\infty} n^{-1} \ln\, \alpha_n(\beta,\theta) = -\tfrac{1}{2}\, c_T(\theta)\,. \tag{1.2.4}$$

Suppose that (1.2.4) fails for some β. Let us define for this β the sequence $\{c_n\}$ by formula (1.1.1). There exist an increasing subsequence $\{n_k\}$ and $\varepsilon > 0$ such that one of the following two inequalities

$$-n_k^{-1} \ln\, \alpha_{n_k}(\beta,\theta) < \tfrac{1}{2}\, c_T(\theta) - \varepsilon\,, \tag{a}$$

$$-n_k^{-1} \ln\, \alpha_{n_k}(\beta,\theta) > \tfrac{1}{2}\, c_T(\theta) + \varepsilon \tag{b}$$

is valid for all k.

In the case (a) we obtain, as a result of (1.2.2), that

$$\begin{aligned} P_\theta\big(-n_k^{-1}\ln L_{n_k} > \tfrac{1}{2}\, c_T(\theta) - \varepsilon\big) &\le P_\theta\big(-n_k^{-1}\ln L_{n_k} > -n_k^{-1}\ln \alpha_{n_k}(\beta,\theta)\big) \\ &= P_\theta\big(L_{n_k} < \alpha_{n_k}(\beta,\theta)\big) \\ &= P_\theta\big(1 - G_{n_k}(T_{n_k}) < 1 - G_{n_k}(c_{n_k})\big) \\ &\le P_\theta\left(T_{n_k} > c_{n_k}\right) \le \beta \qquad \text{for all } k\,. \end{aligned}$$

The left-hand side of the considered inequality tends to 1 as $k \to \infty$, which contradicts the assumption $0 < \beta < 1$. The other case (b) may be examined analogously.

Now let us derive from (1.2.4) the conclusion of the theorem. Put for brevity $N_\alpha := N_T(\alpha, \beta, \theta)$. It follows from (1.2.4) that

$$\alpha_n(\beta,\theta) > \exp\{-n\, c_T(\theta)\}$$

for sufficiently large n. Therefore for any such n the inequality

$$\alpha < \exp\{-n\, c_T(\theta)\}$$

entails $N_\alpha > n$, ensuring that $N_\alpha \to \infty$ as $\alpha \to 0$.

The definition of N_α implies that

$$\alpha_{N_\alpha}(\beta,\theta) \le \alpha \le \alpha_{N_\alpha - 1}(\beta,\theta)$$

or, equivalently,

$$-\,N_\alpha^{-1}\,\ln\,\alpha_{N_\alpha-1}(\beta,\theta) \le -N_\alpha^{-1}\,\ln\,\alpha \le -N_\alpha^{-1}\,\ln\,\alpha_{N_\alpha}(\beta,\theta)\,.$$

The transition to the limit as $\alpha \to 0$ together with (1.2.4) completes the proof of (1.2.3). □

Thus, if two sequences of statistics $\{V_n\}$ and $\{T_n\}$ are such that (1.2.2) holds, their Bahadur ARE $e^B_{V,T}(\beta,\theta)$ can be calculated, according to (1.1.2), by means of the formula

$$e^B_{V,T}(\beta,\theta) = c_V(\theta)\,/\,c_T(\theta)\,. \tag{1.2.5}$$

If $e^B_{V,T} > 1$ for some θ then we should prefer the sequence $\{V_n\}$ to $\{T_n\}$.

The following theorem contains the most simple method for the calculation of exact slopes.

Theorem 1.2.2 (Bahadur 1967, 1971) *Let for a sequence* $\{T_n\}$ *the following two conditions be fulfilled:*

$$T_n \xrightarrow{P_\theta} b(\theta)\,, \qquad\qquad \theta \in \Theta_1\,, \tag{1.2.6}$$

where $-\infty < b(\theta) < \infty$;

$$\lim_{n\to\infty} n^{-1} \ln \left[1 - G_n(t)\right] = -f(t) \tag{1.2.7}$$

for each t *from an open interval* I *on which* f *is continuous and* $\{b(\theta), \theta \in \Theta_1\} \subset I$. *Then (1.2.2) is valid and, moreover, for any* $\theta \in \Theta_1$

$$c_T(\theta) = 2\, f\big(b(\theta)\big)\,. \tag{1.2.8}$$

Formula (1.2.8) plays an exceptionally important role in the Bahadur theory. It shows that for the calculation of exact slopes it is necessary to solve the problem of determining *large-deviation asymptotics* of a sequence $\{T_n\}$ under the null hypothesis. This problem is always nontrivial as Bahadur (1971) himself notes. By contrast the verification of (1.2.6) does not usually present any difficulties.

In the cases when a sequence $\{T_n\}$ does not satisfy the conditions of Theorem 1.2.2, usually one succeeds in selecting strictly monotone functions ψ_n such that a new sequence $\{T_n^*\}, T_n^* := \psi_n(T_n)$, already satisfies these conditions. Since the P-value L_n stays invariable under this transform, the exact slope of $\{T_n\}$ coincides with the exact slope of $\{T_n^*\}$, which might be calculated by (1.2.8).

Proof of Theorem 1.2.2 Fix an arbitrary $\theta \in \Theta_1$ and $\varepsilon > 0$ such that $(b - \varepsilon,\, b + \varepsilon) \subset I$. Put

$$\mathbf{\Omega} := \left\{\, s \in \mathfrak{X}^{(\infty)}\colon\; b - \varepsilon < T_n(s) < b + \varepsilon \,\right\}.$$

It follows from (1.2.6) that for sufficiently large n the estimate $P_\theta\,(\mathbf{\Omega}) > 1 - \delta$ is valid for any $\delta > 0$. As F_n is monotone, the following inequalities

$$1 - F_n(b + \varepsilon) \le L_n(s) \le 1 - F_n(b - \varepsilon)$$

are valid for the same $s \in \mathbf{\Omega}$. Taking logarithms and passing to the limit as $n \to \infty$, we obtain under condition (1.2.7) that

$$-f(b + \varepsilon) \le \varliminf_{n\to\infty} n^{-1} \ln L_n \le \varlimsup_{n\to\infty} n^{-1} \ln L_n \le -f(b - \varepsilon)$$

for each $s \in \mathbf{\Omega}$. By virtue of the continuity of f and of ε being arbitrary it follows now that in P_θ-probability

$$\lim_{n\to\infty} n^{-1} \ln L_n = -f(b)\,. \qquad \square$$

The right-hand side of (1.2.2) may be, generally speaking, a random variable. Note that such situations were discussed by Bahadur and

Raghavachari (1972). It is natural in that case to call the exact slope *stochastic* (see Berk and Brown (1978) and Kallenberg (1981)) as distinct from nonstochastic exact slopes defined by (1.2.4). Fortunately it follows from Theorem 1.2.1 that, if the limit in (1.2.2) is nonrandom, both notions coincide. In the sequel we shall meet only nonrandom limits in (1.2.2), the values of which may be calculated via Theorem 1.2.2.

Another fundamental result in the Bahadur theory is the existence of an upper bound for exact slopes that is sometimes compared in the literature with the Cramér–Rao inequality in the estimation theory.

Define for any two elements P_θ and $P_{\theta'}$ of the basic family of distributions the *Kullback–Leibler information number* (or simply *information*) by means of the formula

$$K(P_\theta, P_{\theta'}) := \begin{cases} \displaystyle\int_{\mathfrak{X}} \ln \frac{d\,P_\theta}{d\,P_{\theta'}}\, d\,P_\theta & \text{if } P_\theta \ll P_{\theta'}\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.9}$$

We shall henceforth often write $K(\theta, \theta')$ instead of $K(P_\theta, P_{\theta'})$. The properties of the Kullback–Leibler information have been examined by Kullback (1959), Bahadur (1971), and Borovkov (1984) among others. It is well known that $K(\theta, \theta') \geq 0$ and $K(\theta, \theta') = 0$ only if $P_\theta = P_{\theta'}$.

Put for any $\theta \in \Theta_1$

$$K(\theta, \Theta_0) := \inf\left\{K(\theta, \theta_0)\colon\ \theta_0 \in \Theta_0\right\}. \tag{1.2.10}$$

Theorem 1.2.3 (Raghavachari 1970; Bahadur 1971) *For any $\theta \in \Theta_1$ with P_θ-probability* 1 *we have*

$$\varliminf_{n\to\infty} n^{-1} \ln L_n(s) \geq -K(\theta, \Theta_0)\,. \tag{1.2.11}$$

Proof Only the case when $K(\theta,\ \Theta_0) < \infty$ is of interest. Fix a $\theta \in \Theta_1$ for which it is valid. For any $\varepsilon > 0$ there exists $\theta_0 \in \Theta_0$ such that

$$0 \leq K(\theta, \theta_0) < K(\theta, \Theta_0) + \varepsilon < +\infty\,. \tag{1.2.12}$$

For any fixed θ and θ_0 denote for brevity $K(\theta, \theta_0)$ by K. If $K < \infty$ we have $P_\theta \ll P_{\theta_0}$ and consequently

$$d\,P_\theta = r(x)\; d\,P_{\theta_0} \qquad \text{on } \left(\mathfrak{X}, \mathfrak{A}\right),$$

$$d\,P_\theta^{(n)} = r_n(s)\; d\,P_{\theta_0}^{(n)} \qquad \text{on } \left(\mathfrak{X}^{(n)}, \mathfrak{A}^{(n)}\right),$$

where $r_n(s) := \Pi_{i=1}^n r(X_i)$. Note that by the strong law of large numbers one can state that with P_θ-probability 1

$$\lim_{n\to\infty} n^{-1} \ln r_n(s) = K\,. \tag{1.2.13}$$

For any positive integer n let us introduce the events

$$A_n := \{L_n < \exp[-n(K+2\varepsilon)]\}\,, \qquad B_n := \{r_n < \exp[n(K+\varepsilon)]\}\,.$$

Then

$$\begin{aligned} P_\theta(A_n B_n) &= \int_{A_n B_n} dP_\theta^{(n)} = \int_{A_n B_n} r_n\, dP_{\theta_0}^{(n)} \\ &\le \exp\{n(K+\varepsilon)\} \int_{A_n} dP_{\theta_0}^{(n)} \\ &= \exp\{n(K+\varepsilon)\} \cdot P_{\theta_0}(A_n) \le \exp\{-n\varepsilon\}\,, \end{aligned} \tag{1.2.14}$$

where the last inequality follows from (1.2.1).

It follows from (1.2.14) that $\sum_n P_\theta(A_n B_n) < \infty$. By the Borel–Cantelli lemma only a finite number of events $A_n B_n$ occurs with P_θ-probability 1. Taking into account (1.2.13) we obtain that the inequality

$$L_n(s) \ge \exp\{-n(K+2\varepsilon)\}$$

holds almost surely for sufficiently large n. Under condition (1.2.12) we establish the conclusion of Theorem 1.2.3 due to the arbitrary choice of ε. Some generalizations are contained in Bahadur and Raghavachari (1972) and Bahadur, Chandra, and Lambert (1982). □

Theorem 1.2.3 implies that the exact slope $c_T(\theta)$ of any sequence of statistics $\{T_n\}$ satisfies the inequality

$$c_T(\theta) \le 2\,K(\theta, \Theta_0)\,. \tag{1.2.15}$$

If equality takes place in (1.2.15) for all $\theta \in \Theta_1$, the sequence $\{T_n\}$ is said to be *asymptotically optimal (AO) in the Bahadur sense.* The class of such statistics is apparently rather narrow, though it contains under certain conditions the likelihood ratio statistics (see Bahadur (1965, 1967) and Rublik (1989)). But if for each $\theta_0 \in \partial\Theta_0$ the weaker condition holds, namely

$$c_T(\theta) \sim 2\,K(\theta, \Theta_0)\,, \qquad \theta \to \theta_0\,, \tag{1.2.16}$$

then the sequence $\{T_n\}$ is said to be *locally asymptotically optimal (LAO) in the Bahadur sense.*

In the initial papers on the Bahadur ARE it was impossible to find the function f in (1.2.7) because of insufficient development of large-deviation theory. In this connection it was proposed by Bahadur (1960b) that the exact distribution of $\{T_n\}$ in the definition of the P-value be replaced by its limiting distribution. Suppose that for all $\theta_0 \in \Theta_0$ and $t \in \mathbf{R}^1$ there exists continuous distribution function F such that

$$F_n(t, \theta_0) \longrightarrow F(t) \qquad \text{as } n \to \infty .$$

Then the substitute of $L_n(s)$ is

$$L_n^*(s) := 1 - F\big(T_n(s)\big) ,$$

and what's more in typical cases there exists the limit in P_θ-probability

$$\lim_{n\to\infty} \left(-n^{-1} \ln L_n^* \right) = \tfrac{1}{2}\, c_T^*(\theta) > 0 . \tag{1.2.17}$$

If (1.2.17) is actually valid, the function $c_T^*(\theta)$ is called the *approximate* (as opposed to *exact*) *slope of the sequence* $\{T_n\}$. The ratio of approximate slopes of two sequences of statistics $\{V_n\}$ and $\{T_n\}$ is called their *approximate Bahadur* ARE and is denoted by $e^{*\,B}_{V,T}(\theta)$.

The method of calculating approximate slopes analogous with Theorem 1.2.2 still applies (see Bahadur (1960b)): *Suppose a sequence* $\{T_n\}$ *satisfies (1.2.6) for some function b, and for some constant a*, $0 < a < \infty$, *the limiting distribution function F satisfies the condition that*

$$\ln\left[1 - F(t)\right] \sim -\tfrac{1}{2}\, a\, t^2 , \qquad t \to \infty . \tag{1.2.18}$$

Then (1.2.17) holds and, besides,

$$c_T^*(\theta) = a\, b^2(\theta) .$$

Approximate slopes are not very reliable as means of comparing tests because monotone transforms of test statistics may lead to entirely different approximate slopes (see, e.g., Groeneboom and Oosterhoff (1977)). But nevertheless they are still used in the statistical literature, mainly for the following reasons:

The approximate Bahadur ARE may be more easily calculated than any other known type of AREs;

the approximate and exact slopes are often locally (as $\theta \to \theta_0$) equivalent, so the approximate ARE gives a notion of the local exact ARE;

the approximate slopes give a simple method for the calculation of the Pitman ARE (for more detail see Section 1.4).