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Positive Harmonic Functions and Diffusion

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To my parents

What is the difference between process and event? A process happens regularly, following a relatively permanent pattern; an event is extraordinary, irregular. A process may be continuous, steady, uniform; events happen suddenly, intermittently, occasionally. Processes are typical; events are unique. A process follows a law, events create a precedent.

Being human is not a solid structure or a string of predictable facts, but an incalculable series of moments and facts. As a process, man may be described biologically; as an event he can only be understood creatively, dramatically.

Abraham Joshua Heschel

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Preface

Over the past 50 years, the body of results concerning questions of existence and characterization of positive harmonic functions for second-order elliptic operators has been nourished by two distinct sources – one rich in analysis, the other less well endowed analytically but amply compensated by generous heapings of probability theory (more specifically, martingales and stopping times). For example, the results appearing in the first seven sections of Chapter 4, which have been developed for the most part over the past decade, have been proved with nary a word about probability, while the results appearing in Chapter 6 have traditionally been formulated and proved using a probabilistic approach. On the other hand, the Martin boundary theory of Chapters 7 and 8 have long been studied by distinct probabilistic and analytic methods. My original intention was to write a monograph which would provide an integrated probabilistic and analytic approach to a host of results and ideas related, at least indirectly, to the existence and/or characterization of positive harmonic functions. When the undertaking was still in its inchoate stages, it became apparent that such a monograph, if executed appropriately, might serve as a graduate text for students working in diffusion processes. This direction also seemed appealing. Indeed, too numerous have been the occasions on which I explained a result or a 'meta-result' to a student or colleague and then found myself at a loss when it came to suggesting a reference text. I hope this book might ground some of the folklore. In the end, then, the book has been written with two intentions in mind.

I have endeavoured to keep the book as self-contained as the dictates of good taste permit. In particular, the book ought to be accessible to the graduate student who is familiar with the standard first-year fare in analysis, probability and partial differential equations at an American university. It also ought to be accessible to analysts who come to the subject without a probabilistic pedigree – if they are willing to do a little spade work in the first two chapters; I hope it might tempt them to explore the probabilistic tack a bit. Monroe Donsker, my first teacher of probability, was fond of saying that a whole class of problems may be formulated and proved in two distinct fashions – one that the probabilists will understand and one that the analysts will understand. He would then add that S. R. S. Varadhan, my thesis adviser, was the practitioner par excellence of the dual approach. The outlook of these two has been a beacon for me; I hope the book reflects that source adequately.

A brief word on the contents is in order. Chapters 1 and 2 provide the necessary background on diffusion processes. These processes are constructed in Chapter 1 and their properties are investigated in Chapter 2. The reader who is familiar with diffusion processes will probably find it useful, none the less, to scan Chapter 1, Sections 1.10–1.13, where the martingale problem is introduced and then quickly superseded by what I call the generalized martingale problem, which turns out to be very convenient for the considerations of this work. These pages will also familiarize the reader with notation that is used throughout the book.

The first part of Chapter 3 culls a few basic results from functional analysis and PDEs. The second part treats an elliptic operator $L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla + V$ on a domain D in the case where D is bounded and possesses a smooth boundary, and where L is uniformly elliptic with coefficients which are smooth up to the boundary. In this setting, a rigorous treatment of the construction of the corresponding semigroup and generator is given; this in turn leads to the theory of the principal eigenvalue.

In Chapter 4, the notion of principal eigenvalue is extended to operators L on arbitrary domains $D \subseteq R^d$ under the assumption that L is uniformly elliptic with smooth coefficients on compact subsets of D. This extension can be characterized in terms of positive harmonic functions and Green's functions, and leads naturally to the classification of operators as subcritical, critical or supercritical. The results given in Sections 4.1–4.7, developed around this theme, are what I call *criticality theory*.

In Sections 4.8 and 4.9, classical results concerning invariant measures and ergodic behavior for diffusion processes are recast in the language of criticality theory. In Section 4.10, the connection between criticality theory and classical spectral theory is made in the case where L is symmetric with respect to some reference measure. Section 4.11 sketches the extension of the results to the case where D is a manifold.

In the one-dimensional case, and in the radially symmetric multi-dimensional case, many calculations may be worked out explicitly; such calculations are carried out in Chapter 5.

Throughout Chapter 6, it is assumed that $V \equiv 0$. In this case, the notions of criticality and subcriticality reduce to those of recurrence and transience, respectively, for diffusion processes. Various techniques are developed to establish criteria for recurrence or transience.

A supercritical operator possesses no positive harmonic functions, while a critical operator possesses a unique positive harmonic function (up to constant multiples). In the subcritical case, the cone of positive harmonic functions may be more interesting. Chapters 7 and 8 treat the structure of the cone $C_L(D)$ of positive harmonic functions for a subcritical operator L on a domain $D \subset R^d$. Chapter 7 begins with a review of the classical analytic theory of the Martin boundary, from which follows the structure of $C_L(D)$, and of the probabilistic theory of h-transforms and conditioned diffusions. The rest of the chapter is devoted to the development of an alternative probabilistic approach to the Martin boundary in terms of what I dub the *exterior harmonic measure boundary*. In Chapter 8, the Martin boundary is calculated explicitly for several classes of operators; in several cases, the calculation is based on the probabilistic approach developed in Chapter 7.

In the case where $V \equiv 0$, an important connection exists between diffusion processes and bounded harmonic functions. This connection is explored in Chapter 9. No such exposition on bounded harmonic functions would be complete without some discussion of Brownian motion on a manifold of negative curvature. However, this topic alone could be the subject of a book. Thus, in the second half of Chapter 9, some familiarity with some rudimentary definitions in differential geometry is assumed. Furthermore, a fundamental comparison result from differential geometry is stated without proof. With the above caveats, complete proofs are presented for the results in this chapter.

Exercises appear at the end of each chapter. In quite a number of them, the reader is asked to supply a proof or a step of a proof that I deemed judicious to omit from the text. Perhaps for that reason, I have been, I confess, too generous with the hints. Historical notes and references also appear at the end of each chapter.

I am indebted to the late Monroe Donsker, and, especially to S. R. S. Varadhan, for their contribution to my mathematics education. I also owe thanks to Rick Durrett, whose stimulating working seminar at UCLA during the 1983–84 academic year first piqued my interest in positive harmonic functions. It is a pleasure to thank Yehuda Pinchover, who has been a patient listener and an unending source of ideas during the writing of this book. Indeed, it was his work on criticality theory, carried out in a purely analytical manner, that crystalized my resolve to write this book. This work was partially supported by the Fund for the Promotion of Research at the Technion. I would like to thank Sylvia Schur and Debbie Miller for their efficient word processing, which spelled a welcome relief for me. Finally, I thank my wife, Jeanette, for her moral support.

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Notation

We use a hierarchical numbering system for equations and statements. A theorem, corollary, lemma, proposition, or example is a 'statement'. The kth equation in section j of chapter i is labeled $(j.k)$ at the place where it occurs and is cited as $(j.k)$ within chapter i, but as $(i.j.k)$ outside chapter i. The kth statement in section j of chapter i is labeled $j.k$ at the place where it occurs and is cited as STATEMENT $j.k$ within chapter i, but as STATEMENT $i.j.k$ outside chapter i.