BOOK III

GENERAL THEORY OF ALGEBRAIC VARIETIES IN PROJECTIVE SPACE

CHAPTER X

ALGEBRAIC VARIETIES

1. Introduction. This volume is concerned with properties of the points of projective space whose coordinates satisfy a set of homogeneous algebraic equations, these points not being treated as individuals, but as members of the aggregate of solutions of the equations.

We begin by selecting the ground field K over which our projective space is to be constructed. We shall confine ourselves to the case in which K is commutative and without characteristic, but we shall not assume that K is algebraically closed unless this requirement is specifically made. We then construct a projective space of n dimensions over K, denoting it by S_n.

In V, § 3, we saw that we could extend the ground field K to any field K^* containing K, and S_n is then extended to a space S_n^* defined over K^*. There are points $(x_0^*, \ldots, x_n^*)$ in S_n^*, where x_i^* $(i = 0, \ldots, n)$ is in K^*, which are not points of the original space S_n. When the ratios of the coordinates x_i^* are all algebraic over K we shall say that $x^* = (x_0^*, \ldots, x_n^*)$ is an algebraic point of S_n (over K). If at least one of the ratios of the coordinates x_i^* is transcendental, we say that x^* is a transcendental point of S_n. In the course of our investigations we shall have to extend the ground field many times, thus introducing points which are algebraic or transcendental over K, and we shall find it convenient to omit a reference to the extension K^* on most of these occasions. We shall therefore be considering *rational points* of S_n (that is, points the ratios of whose coordinates are in K), *algebraic points* of S_n and *transcendental points*. The term *point* without any further qualification will cover these three kinds of point.

A restriction on the fields from which the coordinates of algebraic and transcendental points are chosen is, however, necessary. If x^* is a point which is not rational, the field $K(x^*) = K(x_0^*, \ldots, x_n^*)$ is a proper extension of K, and since it is formed by adjoining a finite number of elements to K, it is an extension of a finite degree of transcendency. The geometrical content of any result which depends only on the properties of this field is not altered if we replace $K(x^*)$ by an equivalent extension of K, and x^* by the corresponding element of this new extension.

But we may have to consider a number (always finite) of extensions of K, and as we saw in III, § 4, p. 114, there may be no extension of K which contains them all. This gives rise to grave difficulties. All the extensions which arise, however, are algebraic extensions of pure transcendental extensions of finite dimension. If K_1 and K_2 are extensions of K, K_2 being of dimension d, we can construct an extension $\bar{K}_1$ of K_1 which contains the algebraic closure of an extension of K of dimension d obtained by adjoining the independent indeterminates $t_1, \ldots, t_d$, and this will contain a field K_2' isomorphic with K_2. But we cannot simply replace K_2 by K_2', for there is ambiguity in the choice of K_2'; for instance, the dimension of the field which is the intersection of K_1 and K_2' depends on the number of the t_i in K_1.

To overcome these difficulties we must ensure that the various extensions of K which we consider all belong to the same extension of K. When this condition is satisfied, the join of the various extensions considered is contained in an extension K^* of a finite degree of transcendency (the enveloping field). Results will be unaltered if K^* is replaced by an equivalent extension of K.

We can lay down a field once and for all over a given ground field K which will contain the isomorph of any enveloping field K^* which may arise, and agree that all extensions be subfields of this. Let $t_1, t_2, \ldots$ be a simple sequence of independent indeterminates over K, Σ the field consisting of rational functions of these (each element of Σ being a rational function of a finite number of t_i), and let Σ^* be the algebraic closure of Σ. Any enveloping field K^* is clearly isomorphic to a subfield of Σ^*, and hence if we replace K^* by an isomorph in Σ^*, each of the extensions in question is replaced by a subfield of Σ^*. We shall call Σ^* the 'universal field' associated with K, and in future we shall assume, without mentioning the fact explicitly, that all extensions of K are in this universal field.

We now choose an allowable coordinate system in S_n, and we consider the set of points whose coordinates satisfy the equations

$$f_i(x_0, x_1, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots), \tag{1}$$

where $f_i(x_0, x_1, \ldots, x_n)$ is a homogeneous polynomial over K. In IV, § 1, we saw that the polynomials over K which are satisfied by the solutions of the set of equations (1) form an ideal, and in IV, § 2, we proved the existence of a finite set of the equations (1), say

$$f_i(x_0, x_1, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, k), \tag{2}$$

which is such that every solution of (2) satisfies the equations (1) for all values of i. Hence in considering the points of S_n which satisfy (1), we need only consider the solutions of the finite set of equations (2).

The aggregate of points defined by a set of equations (1) is called an *algebraic variety*. It may happen, of course, that there are no points satisfying the equations. While this case is of no geometrical interest, it cannot always be avoided in theoretical reasoning; but the statement of theorems will be simpler if, for the present, we assume that the varieties we are considering have at least one point. This point may, of course, be rational, algebraic or transcendental.

An algebraic variety in S_n has been defined in a particular allowable coordinate system. Let us consider what happens to the set of equations (2) when we carry out the allowable transformation of coordinates given by the equations

$$y_i = \sum_{j=0}^{n} a_{ij} x_j \quad (i = 0, \ldots, n),$$

or

$$x_i = \sum_{j=0}^{n} b_{ij} y_j \quad (i = 0, \ldots, n),$$

where (b_{ij}) is the matrix inverse to (a_{ij}).

If $(x'_0, \ldots, x'_n)$ satisfies (2), the coordinates $(y'_0, \ldots, y'_n)$ of the same point in the new coordinate system satisfy the set of equations

$$f_i(\sum_j b_{0j} y_j, \ldots, \sum_j b_{nj} y_j) = 0 \quad (i = 1, \ldots, k), \tag{3}$$

and conversely, if $(y'_0, \ldots, y'_n)$ are the coordinates of a point in the new coordinate system which satisfy (3), the coordinates $(x'_0, \ldots, x'_n)$ of the same point in the original coordinate system satisfy (2). Hence, if an aggregate of points in S_n form an algebraic variety in one coordinate system, they form an algebraic variety in any other allowable coordinate system, although the equations in the two systems may be different. In this sense the definition of an algebraic variety is independent of the coordinate system chosen.

2. Reducible and irreducible varieties. If V_1, V_2 are two algebraic varieties given by the equations

$$f_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r), \tag{1}$$

$$g_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, s) \tag{2}$$

respectively, the points common to V_1 and V_2 satisfy both sets of equations simultaneously, and therefore define a third algebraic variety. We call this aggregate of points the *intersection* of V_1 and V_2, and denote it by the symbol

$$V_1 \wedge V_2.$$

This set of points is the point-set theoretic intersection of the sets V_1, V_2. Evidently

$$V_1 \wedge V_2 = V_2 \wedge V_1.$$

The points which satisfy the set of equations

$$f_i(x_0, \ldots, x_n)\, g_j(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r;\ j = 1, \ldots, s) \tag{3}$$

are those points, and only those, which satisfy either (1) or (2). Points which satisfy (1) or (2) evidently satisfy (3). On the other hand, let $(x'_0, \ldots, x'_n)$ be a point which is not on V_2, say, but which satisfies (3). Then for some value of j

$$g_j(x'_0, \ldots, x'_n) \neq 0.$$

If we consider the equations of the set (3) for which j has this particular value and $i = 1, \ldots, r$, we see that

$$f_i(x'_0, \ldots, x'_n) = 0 \quad (i = 1, \ldots, r).$$

Hence $(x'_0, \ldots, x'_n)$ lies on V_1. Similarly, points satisfying (3) which do not lie on V_1 lie on V_2.

We call the algebraic variety defined by (3) the *sum* of V_1 and V_2 and denote it by the symbol

$$V_1 \dot{+} V_2.$$

We have shown that this symbol defines the point-set theoretic sum of the points in V_1 and the points in V_2. Evidently

$$V_1 \dot{+} V_2 = V_2 \dot{+} V_1.$$

If V_3 is an algebraic variety given by the equations

$$h_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, t),$$

our definitions lead to the following associative and distributive laws, as in point-set theory:

$$(a) \qquad V_1 \wedge (V_2 \wedge V_3) = (V_1 \wedge V_2) \wedge V_3,$$

$$(b) \qquad V_1 \dotplus (V_2 \dotplus V_3) = (V_1 \dotplus V_2) \dotplus V_3,$$

$$(c) \qquad V_1 \wedge (V_2 \dotplus V_3) = V_1 \wedge V_2 \dotplus V_1 \wedge V_3.$$

Now let us suppose that every solution of the equations (1) in any *algebraic* extension of K satisfies (2); that is, that every algebraic point of V_1 lies on V_2. Then by Hilbert's zero-theorem [IV, § 8]

$$[g_i(x_0, \ldots, x_n)]^{\rho_i} = \sum_{j=1}^{r} a_{ij}(x_0, \ldots, x_n) f_j(x_0, \ldots, x_n),$$

where ρ_i is a positive integer, and the $a_{ij}(x_0, \ldots, x_n)$ are forms in $K[x_0, \ldots, x_n]$. Hence $g_i(x_0, \ldots, x_n)$ *vanishes on the variety* V_1, that is, $g_i(x'_0, \ldots, x'_n) = 0$ for *all* points $(x'_0, \ldots, x'_n)$ on V_1, not only for algebraic points. Hence *every* solution of (1) satisfies (2).

We then say 'V_1 lies on V_2', or 'V_1 is contained in V_2', and write

$$V_1 \subseteq V_2,$$

or say 'V_2 contains V_1' and write

$$V_2 \supseteq V_1.$$

If $V_1 \subseteq V_2$ and $V_2 \subseteq V_1$, we must have $V_1 = V_2$. If there are points of which are not on V_1, and V_1 lies on V_2, we write

$$V_1 \subset V_2 \quad \text{or} \quad V_2 \supset V_1.$$

If $V_1 \subseteq V_2$ and $V_2 \subseteq V_3$, then $V_1 \subseteq V_3$. Similarly, if $V_1 \subset V_2$ and $V_2 \subset V_3$, then $V_1 \subset V_3$. Hence the relations $\subseteq$ and $\subset$ are transitive. Equivalently, the relations $\supseteq$ and $\supset$ are transitive.

Again, if

$$V = V_1 \dotplus V_2,$$

then $V \supseteq V_1$ and $V \supseteq V_2$. If there are points of V_1 not on V_2, so that

$$V_1 \nsubseteq V_2,$$

we must have $V \supset V_2$.

It is also clear that $\qquad V_1 \supseteq V_1 \wedge V_2,$

and that, if $\qquad V_1 \nsubseteq V_2,$

then $\qquad V_1 \supset V_1 \wedge V_2.$

With these preliminaries we now introduce the notion of *reducibility*. A variety V is said to be *reducible* if it can be expressed as the sum of two algebraic varieties, each distinct from V; that is, if

$$V = V_1 \dot{+} V_2,$$

where $V_1 \subset V$, $V_2 \subset V$.

If V is not reducible it is said to be *irreducible*.

Lemma. If an irreducible variety V lies in the sum of two varieties V_1 and V_2, then it is contained in one or in the other.

We have

$$V \subseteq V_1 \dot{+} V_2.$$

Then

$$V = V_\wedge (V_1 \dot{+} V_2)$$
$$= V_\wedge V_1 \dot{+} V_\wedge V_2.$$

Since V is irreducible we must have either

$$V_\wedge V_1 = V$$

or

$$V_\wedge V_2 = V,$$

that is, V is contained in V_1 or in V_2.

This lemma is easily extended to the case of r varieties $V_1, V_2, \ldots, V_r$. We use it to prove

THEOREM I. *A necessary and sufficient condition for the reducibility of a variety V is the existence of a product fg of two forms $f(x_0, \ldots, x_n)$ and $g(x_0, \ldots, x_n)$ which vanishes at all points of V without either form having this property.*

We suppose in the first place that V is irreducible and that $fg = 0$ on V. If $f = 0$ defines the variety V_1, and $g = 0$ defines the variety V_2, then $fg = 0$ defines $V_1 \dot{+} V_2$, and

$$V \subseteq V_1 \dot{+} V_2,$$

since $fg = 0$ at all points of V. By the above lemma V is contained in V_1 or in V_2, and so either $f = 0$ or $g = 0$ at all points of V.

Now let us suppose that V is reducible. Then we can construct a product fg of two forms f and g which vanishes on V without either f or g doing so. In fact,

$$V = V_1 \dot{+} V_2,$$

where the varieties V_1 and V_2 are distinct and neither contains the other, by hypothesis. Hence, among the forms defining V_1 there must be at least one, f, say, which does not vanish on V_2.

Similarly, there must be a form g which vanishes on V_2 but not on V_1. The product fg vanishes for all points of V, but neither f nor g vanishes for all points of V. This proves the theorem.

We note that the question of the irreducibility of an algebraic variety depends on the choice of the ground field, and that a variety which is irreducible over K may become reducible when K is replaced by an extension K^*. We illustrate this by considering the algebraic variety V defined over K by a single equation

$$f(x) = f(x_0, \ldots, x_n) = 0.$$

Let us suppose that $f(x)$ is irreducible over K. Then if $g(x)$, $h(x)$ are two forms such that their product vanishes on V, we have, by Hilbert's zero-theorem,

$$[g(x)\,h(x)]^\rho = a(x)\,f(x),$$

where ρ is a positive integer, and $a(x)$ is some form in $K[x]$. By the unique factorisation theorem [I, § 8, Th. II] it follows that $f(x)$, which is irreducible, is a factor of $g(x)$ or of $h(x)$, and hence either $g(x)$ or $h(x)$ vanishes on V. Hence, by Theorem I, V is irreducible.

On the other hand, let us suppose that $f(x)$ is reducible, and that it can be written in the form

$$f(x) = f_1(x)\,f_2(x),$$

where $f_1(x)$, $f_2(x)$ are forms in $K[x]$ *not having a common factor*. Then if V_1, V_2 are the varieties defined, respectively, by the equations

$$f_1(x) = 0$$

and $$f_2(x) = 0,$$

neither variety contains the other. For instance, if $V_1 \subseteq V_2$, we should have, by Hilbert's zero-theorem,

$$[f_2(x)]^\sigma = a(x)\,f_1(x),$$

and hence every irreducible factor of $f_1(x)$ would be a factor of $f_2(x)$, contrary to hypothesis. Hence, since $f(x) = f_1(x)\,f_2(x)$,

$$V = V_1 \dot{+} V_2,$$

where $V_1 \nsubseteq V_2$, $V_2 \nsubseteq V_1$, and so V is reducible.

Now let $f(x)$ be a form in $K[x]$ which is irreducible over K. Then the variety V defined by

$$f(x) = 0$$

is irreducible. But it may happen that there is an extension K^* of

K such that $f(x)$ is reducible in $K^*[x]$. Since K is without characteristic, all the irreducible factors of $f(x)$ over K^* are distinct, and therefore, by what has been proved above, V is reducible over the ground field K^*. An example of a form $f(x)$ with this property in the case $n = 1$ is given by the form $x_0^2 + x_1^2$. When K is the field of real numbers this is irreducible, but it is reducible over the field of complex numbers.

The criterion for reducibility given in Theorem I can be described in another way. We have seen [IV, § 1] that the polynomials in $K[x]$ which vanish on a variety V form an ideal in this ring. Every element of the ideal is a sum of homogeneous polynomials, each of which belongs to the ideal. Such an ideal is called a *homogeneous ideal*. If V is irreducible this homogeneous ideal has the property that if $f(x)$, $g(x)$ are two forms whose product belongs to the ideal, then either $f(x)$ or $g(x)$ belongs to the ideal. A homogeneous ideal with this property is said to be *prime*; thus the polynomials in $K[x_0, \ldots, x_n]$ which vanish on an irreducible algebraic variety form a prime homogeneous ideal, and conversely, a prime homogeneous ideal in $K[x_0, \ldots, x_n]$ defines an irreducible variety.

We remark that an algebraic variety V which is irreducible over K in a given coordinate system is also irreducible in any allowable coordinate system. This follows immediately from the definition of irreducibility, and the remark made at the end of § 1.

As a preliminary to the main theorem of this section we prove

THEOREM II. *A sequence of varieties* $V_1, V_2, \ldots$ *in* S_n, *where* $V_1 \supset V_2 \ldots \supset V_r \supset V_{r+1} \ldots$, *must terminate after a finite number of terms.*

Let the equations of V_1 be

$$f_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r_1),$$

and those of V_2

$$f_i(x_0, \ldots, x_n) = 0 \quad (i = r_1 + 1, \ldots, r_2).$$

Since $V_1 \supset V_2$ we may take the equations of V_2 to be

$$f_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r_2).$$

Similarly, let the equations of V_l be

$$f_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r_l).$$

By Hilbert's basis theorem [IV, § 2] there is a finite integer k such that there exist forms $a_{i1}(x), \ldots, a_{ik}(x)$ with the property that

$$f_i \equiv \sum_{j=1}^{k} a_{ij} f_j$$

for each value of i. Let us suppose that

$$r_{l-1} < k \leqslant r_l.$$

Then, inserting zero polynomials a_{ij} if necessary, we can write

$$f_i \equiv \sum_{j=1}^{r_l} a_{ij} f_j. \tag{4}$$

If there is a variety V_{l+1} in the given sequence, we deduce from (4) that

$$V_{l+1} \supseteq V_l.$$

But this contradicts the hypothesis

$$V_l \supset V_{l+1}.$$

The sequence therefore terminates at V_l.

We can now prove

THEOREM III. *Every algebraic variety V can be expressed as the sum of a finite number of irreducible varieties.*

Let us suppose that the theorem is not true for V. Then V must be reducible, say $V = V_1 \dotplus V_2$, where $V \supset V_1$ and $V \supset V_2$. If the theorem is true for both V_1 and V_2, it is true for V. Hence the theorem is false for either V_1 or V_2. Let us suppose that it is false for V_1. Then V_1 is reducible, and we can write $V_1 = V'_1 \dotplus V'_2$, where $V_1 \supset V'_1$, $V_1 \supset V'_2$. As before, we see that the theorem must be false for either V'_1 or V'_2, say for V'_1. We can repeat this process indefinitely, and obtain an infinite sequence of varieties

$$V \supset V_1 \supset V'_1 \supset \ldots,$$

for each of which the theorem is false. By our previous theorem a strictly descending infinite sequence of varieties does not exist, and we therefore conclude that every algebraic variety V is the sum of a finite number of irreducible varieties.

In the expression

$$V = V_1 \dotplus V_2 \dotplus \ldots \dotplus V_k$$

we may omit any component V_i which is contained in the sum

$$V'_i = V_1 \dotplus V_2 \dotplus \ldots \dotplus V_{i-1} \dotplus V_{i+1} \dotplus \ldots \dotplus V_k$$

of the remaining component varieties. For if $V_i \subseteq V'_i$, then by the lemma proved above, V_i must lie in one of the component varieties whose sum is V'_i, and therefore

$$V = V_i \dotplus V'_i = V'_i.$$

A component V_i which can be omitted in this way is said to be *redundant*. When all redundant components of a sum have been omitted we say that we have a *non-contractible* representation of V as a sum of irreducible varieties.

THEOREM IV. *The representation of an algebraic variety V as a non-contractible sum of irreducible varieties is essentially unique.*

If
$$V = V_1 \dotplus V_2 \dotplus \ldots \dotplus V_k = V'_1 \dotplus V'_2 \dotplus \ldots \dotplus V'_l$$
are two non-contractible representations of V, we prove that $k = l$, and that $V_i = V'_i$ $(i = 1, \ldots, k)$, after the components have been suitably arranged.

It follows from the representation that
$$V_1 \subseteq V = V'_1 \dotplus V'_2 \dotplus \ldots \dotplus V'_l,$$
and therefore, by the lemma above, there exists a value of i $(i \leqslant l)$ such that
$$V_1 \subseteq V'_i.$$
We rearrange the varieties $V'_1, \ldots, V'_l$ so that $i = 1$. Then
$$V_1 \subseteq V'_1.$$
A similar argument shows that for some value of j $(j \leqslant k)$
$$V'_1 \subseteq V_j.$$
Hence
$$V_1 \subseteq V'_1 \subseteq V_j,$$
and therefore
$$V_1 \subseteq V_j.$$
Since the representations are non-contractible it follows that $j = 1$, and since $V_1 \subseteq V'_1 \subseteq V_1$, we must have $V_1 = V'_1$.

In the same way we show that
$$V_2 \subseteq V'_i,$$
where $i \neq 1$, since $V_2 \nsubseteq V_1 = V'_1$. We order the varieties so that $i = 2$, and prove, as above, that $V_2 = V'_2$. The theorem follows by induction.

The varieties $V_1, V_2, \ldots, V_k$ are called the *irreducible components* of V.

3. Generic points of an irreducible variety. A point $\xi = (\xi_0, \ldots, \xi_n)$, where the ξ_i lie in some extension K^* of the ground field K, is said to be a generic point of a variety V (over K) if