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# The Laplacian on a Riemannian Manifold

## An Introduction to Analysis on Manifolds

Steven Rosenberg
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# Introduction

From the basic definitions, differential topology studies the global properties of smooth manifolds, while differential geometry studies both local properties (curvature) and global properties (geodesics). This text studies how differential operators on a smooth manifold reveal deep relationships between the geometry and the topology of the manifold. This is a broad and active area of research, and has been treated in advanced research monographs such as [5], [30], [59]. This book in contrast is aimed at students knowing just the basics of smooth manifold theory, say through Stokes' theorem for differential forms. In particular, no knowledge of differential geometry is assumed.

The goal of the text is an introduction to central topics in analysis on manifolds through the study of Laplacian-type operators on manifolds. The main subjects covered are Hodge theory, heat operators for Laplacians on forms, and the Chern-Gauss-Bonnet theorem in detail. Atiyah-Singer index theory and zeta functions for Laplacians are also covered, although in less detail. The main technique used is the heat flow associated to a Laplacian. The text can be taught in a one year course, and by the conclusion the student should have an appreciation of current research interests in the field.

We now give a brief, quasi-historical overview of these topics, followed by an outline of the book's organization.

The only natural differential operator on a manifold is the exterior derivative $d$ taking $k$-forms to $(k+1)$-forms. This operator is defined purely in terms of the smooth structure. Using $d$, we can define de Rham cohomology groups, the Euler characteristic and the degree of a map of smooth manifolds, all of which give topological information [32]. With some more work, we can reformulate intersection theory in terms of integration of closed forms [11], and so in principle determine the entire real cohomology ring of the manifold.

Once we enter the domain of differential geometry by introducing a Riemannian metric on the manifold, we find a series of differential operators $\Delta^k$, the Laplacians on $k$-forms, associated to the metric. In particular, the Laplacian on functions generalizes the usual Laplacians on $\mathbf{R}^n$ and on the circle.

On a compact manifold, the spectrum $\{\lambda_i^k\}$ of $\Delta^k$ contains both topological and geometric information. In particular, by the Hodge theorem the dimension of the kernel of $\Delta^k$ equals the $k^{\text{th}}$ Betti number, and so the Laplacians determine the Euler characteristic $\chi$.

The geometric information contained in $\Delta^k$ is more difficult to extract. We consider the heat equation $(\partial_t + \Delta^k)\omega = 0$ on $k$-forms with solution given by the heat semigroup $e^{-t\Delta^k}\omega_0$, $\omega_0$ being the initial $k$-form. The behavior of the trace of the heat semigroup, $\mathrm{Tr}(e^{-t\Delta^k}) = \sum_i e^{-\lambda_i^k t}$, as $t \to 0$ is controlled by an infinite sequence of geometric data, starting with the volume of the manifold and the integral of the scalar curvature. This is surprising, since the trace is constructed just from the spectrum of $\Delta^k$.

Now the kernel of $\Delta^k$ controls the behavior of $e^{-t\Delta^k}$ as $t \to \infty$. It turns out that the sum $\sum_k (-1)^k \mathrm{Tr}(e^{-t\Delta^k})$ of the traces of the heat kernels is independent of $t$. The long time behavior of this sum equals the Euler characteristic, while the short time behavior is given by an integral of a complicated curvature expression.

If the dimension of the manifold $M$ is two, this equality of long and short time behavior of the heat flow leads to the Gauss-Bonnet theorem: $\chi(M) = \int_M K\, dA$, where $K$ is the Gaussian curvature and $dA$ is the area element. Note the remarkable fact that the integrand is independent of the Riemannian metric. Of course, there are much simpler proofs of Gauss-Bonnet, but this technique shows that a generalization of Gauss-Bonnet exists in higher dimensions. The explicit determination of the curvature integrand in this generalization, originally due to Chern by other methods, is one of the main results of the text. The proof is a modification of techniques introduced by Getzler around 1985.

The Chern-Gauss-Bonnet theorem, first shown around 1945, can itself be generalized. The Riemannian metric induces Hilbert space structures on the spaces of $k$-forms, and so $d$ has an adjoint $\delta$ taking $(k+1)$-forms to $k$-forms. Recall that the index of an operator $D$ on a Hilbert space is given by $\mathrm{ind}(D) = \dim\ker(D) - \dim\mathrm{coker}(D)$ when the kernel and cokernel are finite dimensional. From Hodge theory, we find that the index of the first order geometric operator $d + \delta$ taking even forms to odd forms is just the Euler characteristic.

This suggests that we look for other geometrically defined operators whose index is a topological invariant. One example is the signature operator, whose index, the signature, is an important topological quantity associated to the middle dimensional cohomology of the manifold. The heat equation approach again gives the signature as the integral of a curvature expression. Even more generally, the Atiyah-Singer index theorem, dating from the early 1960s, shows that the index of any elliptic first order geometric operator $D$ is given by such an integral, even though the index need not have an obvious topological interpretation. Thus we can state the Atiyah-Singer index theorem schematically as $\mathrm{ind}(D_g) = \int_M \mathcal{R}(g)$, where $g$ denotes a Riemannian metric and $\mathcal{R}(g)$ denotes the curvature expression.

The index of these operators will be independent of the Riemannian metric, and so $\int_M \mathcal{R}(g)$ is also metric independent, as in the Gauss-Bonnet theorem. This implies that these particular curvature integrands represent the same cohomology class. As a result, Chern-Weil theory, which constructs representatives of certain cohomology classes from a Riemannian metric, naturally enters the picture.

To summarize, the Gauss-Bonnet theorem equates the topological quantity

$\chi(M)$ with the geometric quantity $\int_M K\ dA$. At the end of the generalization process, the Atiyah-Singer index theorem equates the analytic quantity $\mathrm{ind}(D_g)$ with a topological quantity given by Chern-Weil theory. In fact, the index theorem applies to all elliptic operators, not just geometrically defined operators. This reinterpretation of the nature of both sides of the index theorem indicates the depth of this theory.

To see what lies beyond index theory, we go back to differential topology and first ask what lies beyond cohomology. If the (twisted) cohomology groups of a manifold vanish, a subtler or secondary topological invariant, the Reidemeister torsion, is well defined. In the 1970s, concurrent with the development of the heat equation approach to index theory, Ray and Singer proposed an analytic analogue of Reidemeister torsion defined from the Laplacians on $k$-forms. The definition involves the zeta function of the Laplacian, which encodes the spectrum of the Laplacian differently from the trace of the heat operator. This analytic torsion was shown to equal the Reidemeister torsion around 1980. Recent work of Bismut and Lott [8] has clarified the connection between index theory and analytic torsion.

Index theory and analytic torsion continue to develop in many directions, including K-theory, operator theory, number theory and mathematical physics. (For example, in operator theory it is now considered hopelessly old fashioned to think of the index as an integer.) Hopefully, readers of this book will contribute to this development.

In more detail, Chapter 1 treats the heat equation approach to Hodge theory. We discuss heat flow for the Laplacian on $\mathbf{R}$ and the circle. Riemannian metrics are defined, as are the associated Hilbert spaces of $k$-forms. The Laplacians on forms are given in terms of the Riemannian metric. After proving the basic analytic results (Sobolev embedding theorem, Rellich compactness theorem), we give a heat equation proof of the Hodge theorem, which gives an eigenform decomposition of the Hilbert spaces of forms generalizing Fourier series on the circle. This proof assumes the existence of an integral kernel, the heat kernel, for heat flow for the Laplacians on forms; the construction of the heat kernel is in Chapter 3. The proof shows that the long time behavior of the heat flow is controlled by the kernel of the Laplacian. We then use Gårding's inequality to prove the standard regularity results for the Laplacians and to give the Hodge decomposition of the spaces of smooth forms. (The more standard elliptic/potential theoretic proof of the Hodge theorem is given in the exercises.) We define the de Rham cohomology groups and show that the kernel of the Laplacian on $k$-forms is isomorphic to the $k^{\text{th}}$ de Rham cohomology group. Thus the long time heat flow is controlled by the topology of the manifold.

Chapter 2 covers just those parts of differential geometry needed to construct the heat kernel. We introduce the various curvatures associated to a Riemannian metric and prove that the Riemann curvature tensor is the obstruction to a metric being locally flat. We define the Levi-Civita connection for a Riemannian metric, and prove the Bochner formula relating the Laplacian on forms to this connection and the Riemannian curvature. The Bochner formula is proved using supersymmetry/fermion calculus methods, and it leads to a quick proof of

Gårding's inequality used in Chapter 1. We then turn to the study of geodesics and the exponential map, and give a technical computation of the Laplacian on functions in Riemannian polar coordinates.

In Chapter 3, we construct the heat kernel for the Laplacians on functions and forms. We use Duhamel's formula to motivate the complicated calculations. The construction shows that the short time behavior of the heat flow is determined by the geometry of the Riemannian metric. We also show that the heat kernel on functions is positive.

Chapter 4 discusses the heat equation approach to Atiyah-Singer index theory. The main idea is to compare the long time and short time information in the heat flows. We first show that the Euler characteristic is given by an integral of a curvature expression generalizing the Gauss-Bonnet theorem for surfaces. Following unpublished work of Parker [53], we give a fermion calculus proof of the Chern-Gauss-Bonnet theorem by showing that the integrand is the expected Pfaffian of the curvature. There is a brief discussion of Chern-Weil theory, showing how characteristic classes have representative forms constructed from the curvature of a Riemannian metric. This allows us to give a precise formulation of the Hirzebruch signature theorem. We do not prove this theorem, as the fermion calculus is more involved, but refer the reader to proofs in more advanced texts. We briefly discuss the Hirzebruch-Riemann-Roch theorem; this discussion assumes familiarity with complex geometry and can be omitted. (Since the Dirac operator is even trickier to define and is treated extensively in [5], [30], [59], we do not discuss it at all.) Finally, we define elliptic operators and state the Atiyah-Singer index theorem.

In Chapter 5, we discuss the zeta function of the Laplacian on forms. We show that the poles and special values of the zeta function contain the same information as the short time asymptotics of the heat kernel. For the conformal Laplacian on functions, we show that $\zeta(0)$ is a conformal invariant given by a curvature expression, and that $\zeta'(0)$ is a subtler conformal invariant. We then digress to give Sunada's elegant construction of nonhomeomorphic manifolds whose Laplacians on functions have the same spectrum. We define Reidemeister and analytic torsion, which involves interpreting the determinant of the Laplacian in terms of $\zeta'(0)$. We show that analytic torsion is independent of the Riemannian metric, and so gives a smooth invariant of the manifold. We finish with a (not self-contained) discussion of the recent work of Bismut and Lott, which states precisely in what sense analytic torsion arises as a secondary invariant when no information is available from index theory techniques.

I would like to acknowledge the hospitality of Keio University and the University of Warwick, where much of this text was written. Special thanks are due to Tom Parker for explaining his proof of the Chern-Gauss-Bonnet theorem, to Eric Boeckx for a careful reading of the text, and to the NSF and JSPS for their support. A much different form of support was provided by my wife, Sybil, and an extremely different form by my children, Sam and Selene. This book is dedicated to my family.