Chapter 1

The Laplacian on a Riemannian Manifold

In this chapter we will generalize the Laplacian on Euclidean space to an operator on differential forms on a Riemannian manifold. By a Riemannian manifold, we roughly mean a manifold equipped with a method for measuring lengths of tangent vectors, and hence of curves. Throughout this text, we will concentrate on studying the heat flow associated to these Laplacians. The main result of this chapter, the Hodge theorem, states that the long time behavior of the heat flow is controlled by the topology of the manifold.

In §1.1, the basic examples of heat flow on the one dimensional manifolds S^1 and $\mathbf{R}$ are studied. The heat flow on the circle already contains the basic features of heat flow on a compact manifold, although the circle is too simple topologically and geometrically to really reveal the information contained in the heat flow. In contrast, heat flow on $\mathbf{R}$ is more difficult to study, which indicates why we will restrict attention to compact manifolds. In §1.2, we introduce the notion of a Riemannian metric on a manifold, define the spaces of L^2 functions and forms on a manifold with a Riemannian metric, and introduce the Laplacian associated to the metric. The Hodge theorem is proved in §1.3 by heat equation methods. The kernel of the Laplacian on forms is isomorphic to the de Rham cohomology groups, and hence is a topological invariant. The de Rham cohomology groups are discussed in §1.4, and the isomorphism between the kernel of the Laplacian and de Rham cohomology is shown in §1.5.

Before we start, we note that while the simplest differential operator d/dt on the real line generalizes to the exterior derivative d on a smooth manifold, it is not possible to generalize the second derivative to manifolds without the additional structure of a Riemannian metric. Thus it can be argued that the Laplacian is the simplest, and hence the most basic, differential operator on functions on a Riemannian manifold. Just as the study of the exterior derivative leads to important results, such as de Rham's theorem, relating the smooth structure of the manifold to its underlying topological structure, the study of the Laplacian leads to even deeper results, such as the geometric version of the Atiyah-Singer index theorem, which relates the topology, the smooth structure, and the geometry of a Riemannian manifold.

Notation: Given a smooth map $f: M \to N$ between manifolds, we will denote the differential of f by either df or f_*.

1.1 Basic Examples

Because the theory of the Laplacian on a Riemannian manifold involves some technical preliminaries, we begin by examining some simple examples. In fact, considering the Laplacian and the associated heat flow on just S^1 and $\mathbf{R}$ highlights essential differences between the Laplacian on a compact and on a noncompact manifold.

First, recall that if $T : V \to V$ is a symmetric, nonnegative linear transformation of a finite dimensional inner product space V, then there exists an orthonormal basis of eigenvectors of V with eigenvalues $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. The set $\{\lambda_i\}$ is called the *spectrum* of T, denoted $\sigma(T)$, and is characterized by the property

$$\lambda \notin \sigma(T) \Leftrightarrow (T - \lambda I)^{-1} \text{ exists} \Leftrightarrow \text{Ker}(T - \lambda I) = 0.$$

This eigenvector decomposition of V generalizes to the infinite dimensional case where V is a Hilbert space and T is a compact operator, i.e. an operator such that if $\{v_i\}$ is a bounded sequence in V, then $\{Tv_i\}$ has a convergent subsequence. (For example, any projection onto a finite dimensional subspace is compact, and in fact any compact operator is the norm limit of such finite rank operators.) In this case, the spectral theorem for compact operators says that V again has an orthonormal basis of eigenvectors for T, each eigenspace has only finite multiplicity, and the only (finite or infinite) accumulation point for the set of eigenvalues is zero. In particular, since the absolute values of the eigenvalues are bounded, the operator T is itself bounded. Remember that in infinite dimensions a linear operator may well be unbounded, or equivalently discontinuous.

The spectral theorem for compact operators is an easy generalization of the finite dimensional situation. We want to show that this eigenvector decomposition holds for certain *unbounded* differential operators on compact manifolds. The space V will be some Hilbert space of functions or forms on the manifold. We remark that unbounded operators are only defined on a dense subset of a Hilbert space, and in general one must be very careful to define the domains of such operators and their adjoints correctly. The domains of definition of our unbounded operators are rather easy to construct on compact manifolds, but noncompact manifolds are more difficult to treat. We will follow the standard practice of glossing over these problems, but here are some references for the reader: for unbounded operators in general [70] is quite thorough, while the domains of various Laplacians are stated carefully in [26].

1.1.1 The Laplacian on S^1 and R

The first example to consider is the circle. We set $V = L^2(S^1) = L^2(S^1, \mathbf{C})$, the space of complex valued L^2 functions. (We could just as easily deal with real valued functions, but the notation is a bit more involved.) The simplest differential operator on the circle is of course $d/d\theta$. However, this operator generalizes on a manifold to the exterior derivative $d : \Lambda^0 M \to \Lambda^1 M$, which takes one space to a different space and hence does not have a spectrum; only on one dimensional manifolds can one identify one-forms with functions, e.g. by identifying $f(\theta)d\theta$ with $f(\theta)$. The next simplest operator is the Laplacian or second derivative,

$$\Delta = -\frac{d^2}{d\theta^2},$$

where once and for all we adopt the geometers' convention of placing a minus sign in the definition. We will see that this operator does generalize naturally to an operator on a manifold taking functions again to functions.

Exercise 1: *We might try to consider $d : \Lambda^* \to \Lambda^*$ as an operator taking forms of mixed degree to forms of mixed degree, in which case $\sigma(d)$ is well defined. Show that for any manifold the only eigenvalue is zero, that zero has infinite multiplicity, and that there is an infinite dimensional space of forms which are not in the zero eigenspace. Although we have not yet made Λ^* into a Hilbert space, this shows that a nice spectral decomposition of Λ^* with respect to d does not exist.*

The eigenfunction decomposition of $L^2(S^1)$ is quite well known. An orthonormal basis is given by the trigonometric polynomials $\{e^{in\theta}\}$, $n \in \mathbf{Z}$, and $\Delta e^{in\theta} = n^2 e^{in\theta}$. Thus $L^2(S^1)$ decomposes into eigenspaces with eigenvalues $\{n^2 : n = 0, 1, \ldots\}$ and each eigenspace has multiplicity two, except for the eigenspace of zero, which has multiplicity one. Notice that the eigenfunction decomposition of $f \in L^2$ is given by

$$f = \sum_n a_n e^{in\theta} = \sum_n \langle f, e^{in\theta} \rangle e^{in\theta},$$

with

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{S^1} f(\theta)\overline{g(\theta)}\ d\theta$$

the usual L^2 inner product, which is just the Fourier series decomposition of f. Note also that

$$||e^{in\theta}/n|| \to 0, \text{ but } ||\Delta(e^{in\theta}/n)|| \to \infty$$

as $n \to \infty$, so Δ is unbounded.

The (formal) theory of Fourier series is quite old, dating from around 1825. By the end of the 19th century, Sturm-Liouville theory provided a powerful generalization of Fourier series. This theory typically treats operators of the form

$$D = -\frac{d^2}{dt^2} + A(t)\frac{d}{dt} + B(t),$$

for certain smooth functions A and B, acting on $L^2[-\pi,\pi]$ with certain boundary conditions. In particular, if periodic boundary conditions are imposed, we are working on $L^2(S^1)$. According to this theory, all such operators D give an eigenfunction decomposition of $L^2[-\pi,\pi]$. Moreover, the eigenspaces are finite dimensional, and the eigenvalues $\{\lambda_n\}$ accumulate only at ∞. However, for general A and B it will not be possible to determine the corresponding eigenfunctions and eigenvalues. This situation is typical for Laplacian-type operators on compact manifolds.

We now define and compute the spectrum of $\Delta = -d^2/dt^2$ on $L^2(\mathbf{R})$. (This discussion is more advanced than the circle case, and can be skipped as it is not needed later.) For motivation, choose $\lambda \notin \sigma(D)$, where D is an operator of Sturm-Liouville type. Then not only does $(D-\lambda I)^{-1}$ exist, it is also a bounded operator, since

$$\sigma((D-\lambda I)^{-1}) = \{(\lambda_n - \lambda)^{-1}\}$$

is a bounded set. Moreover, as the reader should check for Δ, by the eigenfunction decomposition it can be shown that the range of $D-\lambda I$ is dense in $L^2(S^1)$, and so $(D-\lambda I)^{-1}$ extends to a bounded operator on all of L^2.

This (I hope) justifies the following definition of the spectrum of an unbounded operator; a complete justification is given by the spectral theorem for unbounded operators.

Definition: *Let D be a symmetric unbounded operator on a Hilbert space H. The* spectrum *of D, $\sigma(D) \subset \mathbf{R}$, is defined by the condition $\lambda \notin \sigma(D)$ iff $(D-\lambda I)^{-1}$ can be extended to a bounded operator on all of H. Equivalently, $\lambda \notin \sigma(D)$ iff (i) $\operatorname{Ker}(D-\lambda I) = 0$, (ii) the image of $D-\lambda I$ is dense, and (iii) on the image, $(D-\lambda I)^{-1}$ is bounded.*

Now set $D = \Delta$ acting on $H = L^2(\mathbf{R})$. We first consider $\lambda \geq 0$. If we look for eigenfunctions, we find that the only solutions to $\Delta f = \lambda f$ are $f(x) = \exp(\pm i\sqrt{\lambda}x)$, for any $\lambda \in \mathbf{R}^+ \cup 0$. However, none of these eigenfunctions is in L^2. This does not mean that $\sigma(\Delta) = \emptyset$. Consider a function $\psi_N(x)$ on $\mathbf{R}$ which satisfies $\psi_N \geq 0$ and

$$\psi_N(x) = \begin{cases} 0 & \text{if } x \in (-\infty, -N] \cup [N, \infty), \\ 1 & \text{if } x \in [-N+1, N-1]. \end{cases}$$

For fixed $\lambda \geq 0$, there exist constants $C, C' > 0$ such that

$$||(\Delta - \lambda I)(\psi_N e^{i\sqrt{\lambda}x})|| \leq C \leq \frac{C'}{N}||\psi_N e^{i\sqrt{\lambda}x}||. \tag{1.1}$$

For ψ_N is nonconstant only on $[-N, -N+1] \cup [N-1, N]$, so the function on the left hand side of (1.1) is zero except on this interval.

Exercise 2: *(i) Show that (1.1) implies* $||(\Delta - \lambda I)^{-1}|| = \infty$.

(ii) Show that $\lambda < 0 \Rightarrow \lambda \notin \sigma(\Delta)$. Hint: Show that there are no L^2 eigenfunctions for λ. If the image of $\Delta - \lambda I$ is not dense, take a non-zero $\alpha \in L^2(\mathbf{R})$ with $\alpha \perp \operatorname{Im}(\Delta - \lambda I)$. Show that

$$0 = \int_{\mathbf{R}} \alpha(x)(\Delta - \lambda I)f(x)$$

for all compactly supported functions f. Thus α is a distributional or weak solution to $(\Delta - \lambda I)\alpha = 0$. (That is, if α were in $L^2(\mathbf{R})$, then we could integrate by parts and let f be a bump function to conclude that $(\Delta - \lambda I)\alpha = 0$.) Classical elliptic regularity results for the Laplacian then imply that in fact $\alpha \in L^2(\mathbf{R})$ (cf. §1.3.4). This is a contradiction. Finally, show that $||(\Delta - \lambda I)^{-1}|| \leq |\lambda|^{-1}$. For this last step, let $g = (\Delta - \lambda I)^{-1} f$. Show that in the L^2 norm and inner product we have

$$||f||^2 = \langle \Delta g, \Delta g\rangle - 2\lambda\langle \Delta g, g\rangle + \lambda^2\langle g, g\rangle \geq \lambda^2||g||^2.$$

Here $\langle \Delta g, g\rangle \geq 0$ by an integration by parts. Thus $||g||^2/||f||^2 \leq (\lambda^2)^{-1}$.

From this exercise, we see that in fact $\sigma(\Delta) = [0, \infty)$.

Since the spectrum does not consist of a discrete set, we cannot have a Fourier series decomposition as on S^1. However, note the analogy between the Fourier series decomposition on S^1,

$$f(\theta) = \sum_n \left(\frac{1}{2\pi}\int_{S^1} f(\psi)e^{-in\psi}\, d\psi\right)e^{in\theta}, \tag{1.2}$$

and the Fourier inversion formula on $\mathbf{R}$,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f(y)e^{-i\xi y}\, dy\right)e^{i\xi x}\, d\xi. \tag{1.3}$$

While we will prove the existence of a formula corresponding to (1.2) on any compact manifold, no such formula corresponding to (1.3) is known for general noncompact manifolds. (In fact, the existence of (1.3) reflects the fact that $\mathbf{R}$ is a semisimple Lie group.)

1.1.2 Heat Flow on S^1 and $\mathbf{R}$

Given an initial distribution $f(\theta) = f(0, \theta)$ of heat on S^1, considered to be perfectly insulated, the distribution $f(t, \theta)$ of heat at time t is (allegedly) governed by the heat equation

$$(\partial_t + \Delta)f(t, \theta) = 0. \tag{1.4}$$

This is quite easy to solve explicitly. If $f(t, \theta) = \sum_n a_n(t)e^{in\theta}$ is the Fourier decomposition for $f(t, \theta)$, with $a_n(0) = a_n$ the n^{th} Fourier coefficient for f, then plugging the Fourier decomposition into (1.4) gives

$$0 = \sum_n (\dot{a}_n(t) + n^2 a_n(t))e^{in\theta}.$$

It follows that $a_n(t) = a_n e^{-n^2 t}$, and so

$$f(t,\theta) = \sum_n e^{-n^2 t} a_n e^{in\theta}.$$

Note that as $t \to \infty$, $f(t,\theta) \to a_0$, which is the average value of f. This fits with our intuition that as $t \to \infty$ the heat should reach a constant equilibrium state; since the circle is insulated, the equilibrium value should be the average of the initial heat distribution.

The situation on $\mathbf{R}$ is as expected more complicated. Given the initial heat distribution $f(0,x) = f(x) \in L^2$, set

$$f(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy. \tag{1.5}$$

Exercise 3: *(i) Verify that for continuous functions f,*

$$(\partial_t + \Delta) f(t,x) = 0 \text{ and } \lim_{t\to 0} f(t,x) = f(x).$$

(ii) Fill in the details in the following derivation for $f(t,x)$. We will assume familiarity with the Fourier transform; see Lemma 1.17. Let $\hat{f}(t,\xi)$ denote the Fourier transform of f in the x variable only. Show that $(\partial_t + \Delta) f(t,x) = 0$ implies

$$-|\xi|^2 \hat{f}(t,\xi) = \partial_t \hat{f}(t,\xi).$$

Conclude that $\hat{f}(t,\xi) = \hat{f}(\xi) e^{-t|\xi|^2}$. Thus

$$\begin{aligned} \hat{f}(t,\xi) &= \hat{f}(\xi) e^{-t|\xi|^2} = \hat{f}(\xi) e^{-|\xi\sqrt{2t}|^2/2} \\ &= \hat{f}(\xi)\{\frac{1}{\sqrt{2t}} e^{-|x|^2/4t}\}\hat{\ }(\xi) \\ &= \{f * (\frac{1}{\sqrt{2t}} e^{-|x|^2/4t})\}\hat{\ }(\xi), \end{aligned}$$

where $\{\ldots\}\hat{\ }$ denotes the Fourier transform of the function in the braces, and the star in the last line denotes convolution. Taking inverse Fourier transform yields

$$f(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-(x-y)^2/4t} f(y)\, dy.$$

(iii) Similarly show that the integral kernel for the heat equation on $\mathbf{R}^n$ is

$$e(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}.$$

Thus (1.5) provides a solution to the heat equation in the form of an integral kernel

$$e(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}, \tag{1.6}$$

which is a smooth function on $(0,\infty)\times\mathbf{R}\times\mathbf{R}$. Notice that for all $f\in L^2$, $f(t,x)\to 0$ as $t\to\infty$, in accordance with our intuition that the heat should "dissipate to $\pm\infty$." While the solution (1.5) looks different from that of S^1, note that on S^1

$$\begin{aligned} f(t,\theta) &= \sum_n e^{-n^2t}\langle f, e^{in\theta}\rangle e^{in\theta} \\ &= \frac{1}{2\pi}\int_{S^1}\sum_n e^{-n^2t}e^{in\theta}\overline{e^{in\psi}}f(\psi)\ d\psi. \end{aligned}$$

Thus heat flow on S^1 is also given by the integral kernel

$$\sum_n e^{-n^2t}e^{in\theta}\overline{e^{in\psi}},$$

which is easily seen to be smooth on $(0,\infty)\times S^1\times S^1$.

Exercise 4: *Prove the Weierstrass approximation theorem: given $f\in C_c(\mathbf{R}^n)$ (where C_c denotes compactly supported continuous functions) and $\epsilon>0$, there exists a polynomial $p(x)$ such that $||f-p||_\infty<\epsilon$ on the support of f. Hint: Apply the heat flow to f, which is valid since $f\in L^2$. Then apply Taylor's theorem to the resulting smooth function.*

These two heat kernels are in fact closely related. First, we have to readjust the inner product on $L^2(S^1)$ by setting $\langle f,g\rangle=\int_{S^1}f(\theta)\overline{g(\theta)}\ d\theta$. (This is just the integration theory on S^1 induced by the local isometry $\mathbf{R}\to S^1$ given by $x\mapsto x \pmod{2\pi}$.) This changes the orthonormal basis to $\{e^{in\theta}/\sqrt{2\pi}\}$, but leaves the rest of the previous discussion unchanged. Let $e_{\mathbf{R}}, e_{S^1}$ denote the heat kernels on $\mathbf{R}, S^1$, respectively. Set

$$\tilde{e}_{S^1}(t,\theta,\psi)=\sum_{n\in\mathbf{Z}}e_{\mathbf{R}}(t,\theta,\psi+2\pi n), \tag{1.7}$$

where on the right side of (1.7) we consider θ,ψ to run over any interval of length 2π in $\mathbf{R}$; this is well defined since $e_{\mathbf{R}}(t,x,y)=e_{\mathbf{R}}(t,x+k,y+k)$ for $k\in\mathbf{R}$. Intuitively, we expect that $\tilde{e}_{S^1}=e_{S^1}$, since heat can "get" from $\theta\in S^1$ to $\psi\in S^1$ by flowing around the circle any number of times in either direction, or equivalently by flowing from $\theta\in\mathbf{R}$ to any translate of ψ in $\mathbf{R}$. This is indeed the case:

Lemma 1.8 $\tilde{e}_{S^1}=e_{S^1}$. *Thus*

$$e_{S^1}(t,\theta,\psi)=\sum_{n\in\mathbf{Z}}e_{\mathbf{R}}(t,\theta,\psi+2\pi n).$$

PROOF. It is immediate that $(\partial_t+\Delta)\tilde{e}_{S^1}=0$, where Δ acts on θ. Also,

$$\lim_{t\to 0}\int_{S^1}\tilde{e}_{S^1}(t,\theta,\psi)f(\psi)\ d\psi=\lim_{t\to 0}\int_{\mathbf{R}}e_{\mathbf{R}}(t,\theta,\psi)f(\psi)\ d\psi,$$

where we extend f to be periodic on $\mathbf{R}$. It is easy to check that even though $f \notin L^2(\mathbf{R})$, we do have

$$\lim_{t\to 0}\int_{\mathbf{R}} e_{\mathbf{R}}(t,\theta,\psi)f(\psi)\ d\psi = f(\theta).$$

The following lemma, whose generalization to manifolds will be used later, finishes the proof of Lemma 1.8:

Lemma 1.9 *Let* $a(t,\theta,\psi), b(t,\theta,\psi) \in C^\infty(\mathbf{R}^+ \times S^1 \times S^1)$ *satisfy* $(\partial_t + \Delta)a = (\partial_t + \Delta)b = 0$, *and*

$$\lim_{t\to 0}\int_{S^1} a(t,\theta,\psi)f(\psi)\ d\psi = \lim_{t\to 0}\int_{S^1} b(t,\theta,\psi)f(\psi)\ d\psi = f(\theta),$$

for all $f \in L^2(S^1)$. *Then* $a(t,\theta,\psi) = b(t,\theta,\psi)$.

Here we make the convention that the Laplacian always acts on the first space variable of the kernel unless otherwise indicated. When necessary, we use the notation $\Delta_\theta, \Delta_\psi$, etc. to indicate on which variable the Laplacian acts.

PROOF. We first show that $a(t,\theta,\psi) = a(t,\psi,\theta)$; similarly, $b(t,\theta,\psi)$ is symmetric in the space variables. For fixed θ,ψ two integrations by parts in the variable μ yield

$$\begin{aligned} 0 &= \int_{S^1} \Delta_\mu a(t',\psi,\mu)\cdot a(t-t',\theta,\mu) - a(t',\psi,\mu)\cdot \Delta_\mu a(t-t',\theta,\mu) \\ &= \int_{S^1} -\partial_{t'} a(t',\psi,\mu)\cdot a(t-t',\theta,\mu) - a(t',\psi,\mu)\cdot \partial_{t'} a(t-t',\theta,\mu) \\ &= -\partial_{t'}\int_{S^1} a(t',\psi,\mu)\cdot a(t-t',\theta,\mu). \end{aligned}$$

Thus if we abbreviate $\lim_{t\to 0}\int_{S^1} a(t,\theta,\mu)f(\mu)$ by $\int_{S^1} a(0,\theta,\mu)f(\mu)$, we have

$$\begin{aligned} 0 &= \int_0^t dt'\ \partial_{t'}\int_{S^1} a(t',\psi,\mu)\cdot a(t-t',\theta,\mu) \\ &= \int_{S^1} a(t,\psi,\mu)\cdot a(0,\theta,\mu) - \int_{S^1} a(0,\psi,\mu)\cdot a(t,\theta,\mu) \\ &= a(t,\psi,\theta) - a(t,\theta,\psi). \end{aligned}$$

Now consider the integral

$$\int_0^t ds\ \partial_s \int_{S^1} a(s,\theta,\mu)b(t-s,\mu,\psi)\ d\mu. \tag{1.10}$$

This equals

$$\lim_{t'\to 0}\left[\int_{S^1} a(t-t',\theta,\mu)b(t',\mu,\psi)\ d\mu - \int_{S^1} a(t',\theta,\mu)b(t-t',\mu,\psi)\ d\mu\right],$$

which reduces to

$$a(t,\theta,\psi) - b(t,\theta,\psi).$$

On the other hand, (1.10) also equals

$$\int_0^t ds\left[\int_{S^1} \partial_s a(s,\theta,\mu)\cdot b(t-s,\mu,\psi)\,d\mu + \int_{S^1} a(s,\theta,\mu)\cdot\partial_s b(t-s,\mu,\psi)\,d\mu\right]$$

$$= \int_0^t ds\left[\int_{S^1} -\Delta_\theta a(s,\theta,\mu)\cdot b(t-s,\mu,\psi)\,d\mu + \int_{S^1} a(s,\theta,\mu)\cdot\Delta_\mu b(t-s,\mu,\psi)\,d\mu\right]. \tag{1.11}$$

By the symmetry of $a(t,x,y)$ in x and y, we can replace $-\Delta_\theta a(s,\theta,\mu)|_{(s,\theta,\mu)}$ in (1.11) by $-\Delta_\mu a(s,\theta,\mu)|_{(s,\mu,\theta)}$. Two integrations by parts then replace the first integrand on the right hand side of (1.11) by

$$-a(s,\mu,\theta)\Delta_\mu b(t-s,\mu,\psi) = -a(s,\theta,\mu)\Delta_\mu b(t-s,\mu,\psi),$$

which cancels with the second integrand, finishing the proof.

We denote the operator taking the heat distribution f to the time t heat distribution $f(t,x)$ by

$$e^{-t\Delta} : L^2 \to L^2,$$

for heat flow on either S^1 or $\mathbf{R}$; the notation is suggested by the fact that $e^{-t\Delta}$ acts by multiplication by e^{-tn^2} on the n^2-eigenspace of Δ on S^1, and is justified by the spectral theorem for unbounded operators in the case of $\mathbf{R}$ (see [70]). The trace of the heat operator on S^1 is given by the "trace" of the heat kernel:

$$\mathrm{Tr}\ e^{-t\Delta} = \sum_n e^{-n^2t} = \int_{S^1} e_{S^1}(t,\theta,\theta)\,d\theta.$$

For short time, the trace of the heat kernel on S^1 looks like the trace of the heat kernel on $\mathbf{R}$, in the sense that

$$\begin{aligned}\sum_n e^{-n^2t} &= \int_{S^1} e_{S^1}(t,\theta,\theta)\,d\theta = \int_{S^1}\sum_n e_{\mathbf{R}}(t,\theta,\theta+2\pi n)\,d\theta \\ &= \int_{-\pi}^{\pi} e_{\mathbf{R}}(t,x,x)\,dx + \mathrm{O}(t^\infty),\end{aligned}$$

where $\mathrm{O}(t^\infty)$ denotes terms dying like $e^{-\frac{\alpha}{t}}$, for some $\alpha > 0$, as $t \to 0$. Thus as $t \to 0$, $\mathrm{Tr}\ e^{-t\Delta}$ on S^1 looks more and more like $2\pi/\sqrt{4\pi t}$. To be precise, we make the following definition:

Definition: *Given functions $A(t)$ and $B(t)$, we write $A(t) \sim B(t)$ if*

$$\lim_{t\to 0}\frac{A(t)-B(t)}{t^m} = 0,$$

for all $m \in \mathbf{R}^+$.

In this notation, we have shown the following result of Jacobi (ca. 1780):

Theorem 1.12

$$\sum_{n\in\mathbf{Z}} e^{-n^2t} \sim \sqrt{\pi}\, t^{-\frac{1}{2}}.$$

In the course of this argument, we have seen that the short time behavior of the pointwise trace of the heat kernel $e_{S^1}(t,\theta,\theta)$ is the same as that of $e_{\mathbf{R}}(t,x,x)$, up to exponentially small factors. This is more or less plausible, if one argues that S^1 and $\mathbf{R}$ are locally isometric, and that for $t \approx 0$, all the information in the heat flow should be locally computable, as the heat has not "had enough time" to distribute itself around the manifold.

Moreover, if we take a circle of circumference ℓ, it is easy to check that $\operatorname{Tr} e^{-t\Delta} \sim \ell/\sqrt{4\pi t}$, so the short time behavior of the heat operator recaptures the length, the only intrinsic geometric invariant of a circle (i.e. any two circles of the same length are isometric as metric spaces). In contrast, the long time behavior of the heat flow clearly differs on S^1 and $\mathbf{R}$, at least to the extent of distinguishing between a compact and a noncompact manifold.

These remarks are the simplest examples of quite general phenomena. We will see in this chapter that the long time behavior of the heat flow on functions and forms is determined by the topology of a compact manifold, and in Chapter 3 we will show that the short time behavior is controlled by the local differential geometry of the manifold. Moreover, comparing the long and short time behavior of the heat flow will lead in Chapter 4 to a proof of the Atiyah-Singer index theorem.

Finally, we should point out that it is unclear how closely the heat equation models actual heat flow. For example, the reader should check that (1.5) shows that $f(t,x)$ is a smooth function in x for any $t > 0$. Is it physically plausible that a discontinuous initial heat distribution in $L^2(\mathbf{R})$, such as a step function, should be immediately smoothed under heat flow, or is such an initial distribution physically implausible? Moreover, from (1.5) we see that for any $t > 0$, the heat distribution at x, namely $f(t,x)$, is affected by the initial distribution $f(y)$ at y, for y arbitrarily far from x. Thus, we say that heat flow has infinite propagation speed, in contrast to the solution of the wave equation. Is this physically plausible?

1.2 The Laplacian on a Riemannian Manifold

The goal of this section is to generalize the notion of the Laplacian on $L^2(S^1)$ to the Laplacian on L^2 functions on any manifold. To do this, we need to introduce a Riemannian metric on the manifold.

1.2.1 Riemannian Metrics

Given a smooth manifold, there is no natural way to define a generalization of the Laplacian on S^1 or on $\mathbf{R}$, without as additional data a "geometry" in the form of a Riemannian metric.