LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor J.W.S. Cassels, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

The titles below are available from booksellers, or, in case of difficulty, from Cambridge University Press.

34 Representation theory of Lie groups, M.F. ATIYAH *et al*
36 Homological group theory, C.T.C. WALL (ed)
39 Affine sets and affine groups, D.G. NORTHCOTT
46 p-adic analysis: a short course on recent work, N. KOBLITZ
50 Commutator calculus and groups of homotopy classes, H.J. BAUES
59 Applicable differential geometry, M. CRAMPIN & F.A.E. PIRANI
66 Several complex variables and complex manifolds II, M.J. FIELD
69 Representation theory, I.M. GELFAND *et al*
74 Symmetric designs: an algebraic approach, E.S. LANDER
76 Spectral theory of linear differential operators and comparison algebras, H.O. CORDES
77 Isolated singular points on complete intersections, E.J.N. LOOIJENGA
79 Probability, statistics and analysis, J.F.C. KINGMAN & G.E.H. REUTER (eds)
83 Homogeneous structures on Riemannian manifolds, F. TRICERRI & L. VANHECKE
86 Topological topics, I.M. JAMES (ed)
87 Surveys in set theory, A.R.D. MATHIAS (ed)
88 FPF ring theory, C. FAITH & S. PAGE
89 An F-space sampler, N.J. KALTON, N.T. PECK & J.W. ROBERTS
90 Polytopes and symmetry, S.A. ROBERTSON
91 Classgroups of group rings, M.J. TAYLOR
92 Representation of rings over skew fields, A.H. SCHOFIELD
93 Aspects of topology, I.M. JAMES & E.H. KRONHEIMER (eds)
94 Representations of general linear groups, G.D. JAMES
95 Low-dimensional topology 1982, R.A. FENN (ed)
96 Diophantine equations over function fields, R.C. MASON
97 Varieties of constructive mathematics, D.S. BRIDGES & F. RICHMAN
98 Localization in Noetherian rings, A.V. JATEGAONKAR
99 Methods of differential geometry in algebraic topology, M. KAROUBI & C. LERUSTE
100 Stopping time techniques for analysts and probabilists, L. EGGHE
101 Groups and geometry, ROGER C. LYNDON
103 Surveys in combinatorics 1985, I. ANDERSON (ed)
104 Elliptic structures on 3-manifolds, C.B. THOMAS
105 A local spectral theory for closed operators, I. ERDELYI & WANG SHENGWANG
106 Syzygies, E.G. EVANS & P. GRIFFITH
107 Compactification of Siegel moduli schemes, C-L. CHAI
108 Some topics in graph theory, H.P. YAP
109 Diophantine analysis, J. LOXTON & A. VAN DER POORTEN (eds)
110 An introduction to surreal numbers, H. GONSHOR
113 Lectures on the asymptotic theory of ideals, D. REES
114 Lectures on Bochner-Riesz means, K.M. DAVIS & Y-C. CHANG
115 An introduction to independence for analysts, H.G. DALES & W.H. WOODIN
116 Representations of algebras, P.J. WEBB (ed)
117 Homotopy theory, E. REES & J.D.S. JONES (eds)
118 Skew linear groups, M. SHIRVANI & B. WEHRFRITZ
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
122 Non-classical continuum mechanics, R.J. KNOPS & A.A. LACEY (eds)
124 Lie groupoids and Lie algebroids in differential geometry, K. MACKENZIE
125 Commutator theory for congruence modular varieties, R. FREESE & R. MCKENZIE
126 Van der Corput's method of exponential sums, S.W. GRAHAM & G. KOLESNIK
127 New directions in dynamical systems, T.J. BEDFORD & J.W. SWIFT (eds)
128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
129 The subgroup structure of the finite classical groups, P.B. KLEIDMAN & M.W.LIEBECK
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
132 Whitehead groups of finite groups, ROBERT OLIVER
133 Linear algebraic monoids, MOHAN S. PUTCHA
134 Number theory and dynamical systems, M. DODSON & J. VICKERS (eds)
135 Operator algebras and applications, 1, D. EVANS & M. TAKESAKI (eds)
136 Operator algebras and applications, 2, D. EVANS & M. TAKESAKI (eds)
137 Analysis at Urbana, I, E. BERKSON, T. PECK, & J. UHL (eds)

138 Analysis at Urbana, II, E. BERKSON, T. PECK, & J. UHL (eds)
139 Advances in homotopy theory, S. SALAMON, B. STEER & W. SUTHERLAND (eds)
140 Geometric aspects of Banach spaces, E.M. PEINADOR and A. RODES (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
142 The geometry of jet bundles, D.J. SAUNDERS
143 The ergodic theory of discrete groups, PETER J. NICHOLLS
144 Introduction to uniform spaces, I.M. JAMES
145 Homological questions in local algebra, JAN R. STROOKER
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
147 Continuous and discrete modules, S.H. MOHAMED & B.J. MÜLLER
148 Helices and vector bundles, A.N. RUDAKOV *et al*
149 Solitons, nonlinear evolution equations and inverse scattering, M. ABLOWITZ & P. CLARKSON
150 Geometry of low-dimensional manifolds 1, S. DONALDSON & C.B. THOMAS (eds)
151 Geometry of low-dimensional manifolds 2, S. DONALDSON & C.B. THOMAS (eds)
152 Oligomorphic permutation groups, P. CAMERON
153 L-functions and arithmetic, J. COATES & M.J. TAYLOR (eds)
154 Number theory and cryptography, J. LOXTON (ed)
155 Classification theories of polarized varieties, TAKAO FUJITA
156 Twistors in mathematics and physics, T.N. BAILEY & R.J. BASTON (eds)
157 Analytic pro-*p* groups, J.D. DIXON, M.P.F. DU SAUTOY, A. MANN & D. SEGAL
158 Geometry of Banach spaces, P.F.X. MÜLLER & W. SCHACHERMAYER (eds)
159 Groups St Andrews 1989 volume 1, C.M. CAMPBELL & E.F. ROBERTSON (eds)
160 Groups St Andrews 1989 volume 2, C.M. CAMPBELL & E.F. ROBERTSON (eds)
161 Lectures on block theory, BURKHARD KÜLSHAMMER
162 Harmonic analysis and representation theory, A. FIGA-TALAMANCA & C. NEBBIA
163 Topics in varieties of group representations, S.M. VOVSI
164 Quasi-symmetric designs, M.S. SHRIKANDE & S.S. SANE
165 Groups, combinatorics & geometry, M.W. LIEBECK & J. SAXL (eds)
166 Surveys in combinatorics, 1991, A.D. KEEDWELL (ed)
167 Stochastic analysis, M.T. BARLOW & N.H. BINGHAM (eds)
168 Representations of algebras, H. TACHIKAWA & S. BRENNER (eds)
169 Boolean function complexity, M.S. PATERSON (ed)
170 Manifolds with singularities and the Adams-Novikov spectral sequence, B. BOTVINNIK
171 Squares, A.R. RAJWADE
172 Algebraic varieties, GEORGE R. KEMPF
173 Discrete groups and geometry, W.J. HARVEY & C. MACLACHLAN (eds)
174 Lectures on mechanics, J.E. MARSDEN
175 Adams memorial symposium on algebraic topology 1, N. RAY & G. WALKER (eds)
176 Adams memorial symposium on algebraic topology 2, N. RAY & G. WALKER (eds)
177 Applications of categories in computer science, M. FOURMAN, P. JOHNSTONE, & A.. PITTS (eds)
178 Lower K- and L-theory, A. RANICKI
179 Complex projective geometry, G. ELLINGSRUD *et al*
180 Lectures on ergodic theory and Pesin theory on compact manifolds, M. POLLICOTT
181 Geometric group theory I, G.A. NIBLO & M.A. ROLLER (eds)
182 Geometric group theory II, G.A. NIBLO & M.A. ROLLER (eds)
183 Shintani zeta functions, A. YUKIE
184 Arithmetical functions, W. SCHWARZ & J. SPILKER
185 Representations of solvable groups, O. MANZ & T.R. WOLF
186 Complexity: knots, colourings and counting, D.J.A. WELSH
187 Surveys in combinatorics, 1993, K. WALKER (ed)
189 Locally presentable and accessible categories, J. ADAMEK & J. ROSICKY
190 Polynomial invariants of finite groups, D.J. BENSON
191 Finite geometry and combinatorics, F. DE CLERCK *et al*
192 Symplectic geometry, D. SALAMON (ed)
193 Computer algebra and differential equations, E. TOURNIER (ed)
195 Arithmetic of blowup algebras, WOLMER VASCONCELOS
196 Microlocal analysis for differential operators, A. GRIGIS & J. SJÖSTRAND
197 Two-dimensional homotopy and combinatorial group theory, C. HOG-ANGELONI, W. METZLER & A.J. SIERADSKI (eds)
198 The algebraic characterization of geometric 4-manifolds, J.A. HILLMAN
199 Invariant potential theory in the unit ball of $\mathbb{C}^n$, MANFRED STOLL

London Mathematical Society Lecture Note Series. 199

Invariant Potential Theory in the Unit Ball of $\mathbb{C}^n$

Manfred Stoll
University of South Carolina

CAMBRIDGE
UNIVERSITY PRESS

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1994

Library of Congress cataloguing in publication data available

British Library cataloguing in publication data available

ISBN 0 521 46830 2 paperback

Transferred to digital printing 2003

Zum Andenken

an

Wladislaw Kobryn und Georg Stoll

Cambridge University Press
0521468302 - Invariant Potential Theory in the Unit Ball of Cn
Manfred Stoll
Frontmatter
More information

Contents

Preface

Introduction 1

1. Notation and Preliminary Results 7
- 1.1 Notation 7
- 1.2 Integral Formulas on B 10
- 1.3 Automorphisms of B 11

2. The Bergman Kernel 12
- 2.1 The Bergman Kernel 12
- 2.2 Examples 16
- 2.3 Properties of the Bergman Kernel 19
- 2.4 The Bergman Metric 20

3. The Laplace-Beltrami Operator 23
- 3.1 The Invariant Laplacian 23
- 3.2 The Invariant Laplacian for U^n 24
- 3.3 The Invariant Laplacian for B 25
- 3.4 The Invariant Gradient 27

4. Invariant Harmonic and Subharmonic Functions 31
- 4.1 $\mathcal{M}$-Subharmonic Functions 31
- 4.2 The Invariant Convolution on B 34
- 4.3 The Riesz Measure 36
- 4.4 Remarks 40

5. Poisson-Szegö Integrals 43
- 5.1 The Poisson-Szegö Kernel 43
- 5.2 The Dirichlet Problem for $\widetilde{\Delta}$ 48
- 5.3 Poisson-Szegö Integrals 51
- 5.4 The Dirichlet Problem for rB 55
- 5.5 Remarks 57

6. The Riesz Decomposition Theorem 60
- 6.1 Harmonic Majorants for $\mathcal{M}$-Subharmonic Functions 61
- 6.2 The Green's Function for $\widetilde{\Delta}$ 64
- 6.3 The Riesz Decomposition Theorem 67

6.4 Green Potentials 71
6.5 A Characterization of H^p Spaces 75
6.6 Remarks 79

7. Admissible Boundary Limits of Poisson Integrals 81

7.1 Admissible Limits of Poisson Integrals 82
7.2 Maximal Functions of Measures 83
7.3 Differentiation Theorems 87
7.4 The Admissible Maximal Function 89
7.5 Weighted Radial Limits of Poisson Integrals 92
7.6 Remarks 94

8. Radial and Admissible Boundary Limits of Potentials 96

8.1 Radial Limits of Potentials 96
8.2 Admissible Limits of Potentials 107
8.3 Tangential Limits of Potentials 114
8.4 Convergence in L^p 120
8.5 Related Results 123

9. Gradient Estimates and Riesz Potentials 126

9.1 Gradient Estimates of Green Potentials 126
9.2 L^p Inequalities for the Riesz Operator 132

10. Spaces of Invariant Harmonic Functions 142

10.1 Mean Value Inequalities for $|h|^p$ and $|\widetilde{\nabla} h|^p$, $0 < p < \infty$ 142
10.2 On a Theorem of Hardy and Littlewood 149
10.3 $\mathcal{M}$-Harmonic Bergman and Dirichlet Spaces 152
10.4 Remarks 160

References 164

Index 171

Preface

These notes are based on a year long seminar given by the author at the University of South Carolina during the 1992-93 academic year. The main purpose of the notes is to introduce the reader to some of the recent results in potential theory with respect to the Laplace-Beltrami operator in several complex variables, with special emphasis on the unit ball in $\mathbb{C}^n$.

The term "invariant" in the title stems from the fact that the Laplace-Beltrami operator $\widetilde{\Delta}$ of a domain is invariant under the biholomorphic mappings of the domain onto itself. Specifically, $\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$ for all biholomorphic mappings ψ of the domain. Potential theory with respect to the Laplace-Beltrami operator on the ball is one of the natural extensions to several complex variables of potential theory in the unit disc. This approach is related to the non-euclidean geometry of the ball, and differs significantly from the usual euclidean extension of potential theory to the unit ball in $\mathbb{R}^n$, $n \geq 3$.

The extension to invariant potential theory in the unit ball in $\mathbb{C}^n$ for $n > 1$ does not just involve the extension of one variable techniques and results to several variables. There are significant difference between $n = 1$ and $n > 1$. Many of the classical results which are true when $n = 1$, fail to be true for the invariant Laplacian when $n > 1$. Since invariant harmonic functions are not preserved under dilations, results in the unit disc or the unit ball in $\mathbb{R}^n$, which used dilations of the domain in their arguments, require new techniques and approaches in the invariant setting. Also, since the singularities of the invariant Poisson kernel and Green's functions are non-euclidean, proofs and statements of results have a greater dependence on the non-euclidean geometry of the domain.

The study of harmonic function theory with respect to the invariant Laplacian $\widetilde{\Delta}$ gained momentum in the 1960's with the Poisson integral representation of bounded harmonic functions on symmetric spaces by Furstenberg, and the extension of the classical Fatou theorem to Poisson integrals on the ball and bounded symmetric domains by Koranyi, Stein, and Weiss. Since that time, there has been considerable research activity in the study of invariant harmonic functions, and more recently, in invariant potential theory in general. Although proofs of many of the results are available elsewhere, the purpose of the notes is to provide a cohesive treatment of this important subject. It is hoped that these notes will not only be useful in providing information on the subject area, but will also stimulate additional research, especially on other domains.

Although our primary emphasis is on potential theory with respect to the invariant Laplacian $\widetilde{\Delta}$ on the unit ball B in $\mathbb{C}^n$, the introduction provides an overview of the various extensions of classical potential theory to several complex variables. Throughout the notes I have attempted to provide references to some of the known results in various other settings, and also to additional results on the ball, the proofs of which have not been included in the notes.

Chapters 2 and 3 deal with general results concerning the Bergman kernel, the Laplace-Beltrami operator, and the invariant gradient on bounded domains in $\mathbb{C}^n$. In Chapters 4 through 6 we include some of the basic properties of functions harmonic and subharmonic with respect to the invariant Laplacian in the unit ball B, including the invariant Poisson kernel, Poisson integrals, and the Riesz decomposition theorem. Chapters 7 and 8 deal with the extension of the classical Fatou's theorem on nontangential limits of Poisson integrals, and Littlewood's theorem on the existence of radial limits of subharmonic functions. In Chapter 8 we also include results on admissible boundary limits of subharmonic functions, and tangential boundary limits of potentials. Chapter 9 contains recent results involving L^p estimates for the invariant gradient of Green potentials, and in Chapter 10 we include several results on weighted Bergman and Dirichlet type spaces of invariant harmonic functions on B.

With a few exceptions, the notes are self contained, and should be accessible to anyone who has some basic knowledge of several complex variables, measure theory, and functional analysis. Most of the references to the preliminary material in several complex variables come from the texts "Function Theory of Several Complex Variables" by Steven Krantz, and "Function Theory in the Unit Ball of $\mathbb{C}^n$" by Walter Rudin. A standard graduate course in real analysis should be sufficient for the prerequisites in measure theory and functional analyis.

The author would like to thank both the faculty and graduate students who participated in the seminar for their patience, endurance, and the many helpful comments they provided. Special thanks go to my students K. Adzievski and L. Rzepecki who proof-read the manuscript and corrected many of my typing errors. Finally, the author would like to apologize to the many researchers studying potential theory in $\mathbb{R}^n$ for neglecting to make reference to their many contributions to the subject area.