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Introduction

In the study of one complex variable, a real valued C^2 function f defined on a domain Ω in $\mathbb{C}$ is harmonic if $\Delta f = 0$, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z \partial \overline{z}}$$

is the Laplacian in $\mathbb{C}$ or $\mathbb{R}^2$. In the above,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It is well known that f is harmonic in Ω if and only if locally f is the real part of a holomorphic function.

In addition to the usual Laplacian on Ω, there is also the invariant Laplacian or the Laplace-Beltrami operator $\widetilde{\Delta}$ which is defined in terms of the Bergman kernel function of Ω. For the unit disc U, this operator is given by

$$\widetilde{\Delta} = 2(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}}.$$

The operator $\widetilde{\Delta}$ has the property that

$$\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$$

for all automorphisms ψ of U. It is clear that $\widetilde{\Delta} f = 0$ if and only if $\Delta f = 0$, and thus when $n = 1$, the euclidean and noneuclidean definitions coincide.

When one considers $\mathbb{C}^n$, $n > 1$, there are many concepts of harmonic, and as a general rule, these are all different. The extensions to $\mathbb{C}^n$ are usually separated into local definitions and global definitions. For a domain Ω in $\mathbb{C}^n$, $n > 1$, the three common local definitions are as follows:

(1) A C^2 function $f : \Omega \to \mathbb{R}$ is **harmonic** (or euclidean harmonic) if $\Delta f = 0$ in Ω, where here

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j}$$

is the usual Laplacian on $\mathbb{R}^{2n}$. It is well known that this is equivalent to the following: a continuous function $f : \Omega \to \mathbb{R}$ is harmonic if for every $a \in \Omega$,

$$f(a) = \int_S f(a + rt)\, d\sigma(t)$$

for all $r > 0$ sufficiently small, where S is the unit sphere in $\mathbb{C}^n$ ($\mathbb{R}^{2n}$) and σ is the normalized rotation invariant measure on S.

(2) A C^2 function $f : \Omega \to \mathbb{R}$ is **n-harmonic** if

$$\frac{\partial^2 f}{\partial z_j \partial \overline{z}_j} = 0 \qquad \text{for all} \qquad j = 1, \dots, n.$$

Equivalently, f is n-harmonic if f is harmonic in each variable separately.

(3) A function $f : \Omega \to \mathbb{R}$ is **pluriharmonic** if for each $a \in \Omega$, $b \in \mathbb{C}^n$, the function

$$\lambda \to f(a + \lambda b)$$

is harmonic in a neighborhood of 0 in $\mathbb{C}$. It is easily shown that a C^2 function f is pluriharmonic if and only if

$$\frac{\partial^2}{\partial z_j \partial \overline{z}_k} = 0$$

for all j, k. This is equivalent to f being locally the real part of a holomorphic function.

From the definitions it is clear that

$$(3) \;\subset\; (2) \;\subset\; (1)\,,$$

and that all containments are proper.

In addition to the above, there are also global definitions which depend on the domain. If Ω is a domain in $\mathbb{C}^n$, we let $Aut(\Omega)$ denote the group of biholomrphic mappings of Ω onto itself. The domain Ω is said to be **homogeneous** if $Aut(\Omega)$ is transitive on Ω, i.e., for z_1, $z_2 \in \Omega$, there exists $\psi \in Aut(\Omega)$ such that $\psi(z_1) = z_2$. A homogeneous domain Ω is **symmetric** if for any $z_o \in \Omega$, there exists $\psi \in Aut(\Omega)$ such that

(a) $\psi(z_o) = z_o$ but $\psi(z) \neq z$ for all $z \neq z_o$, and

(b) $\psi \circ \psi =$ identity on Ω.

Both the unit ball B_n and the unit polydisc U^n are bounded symmetric domains. In $\mathbb{C}^2$, every bounded homogeneous domain is symmetric and is biholomorphic to either B_2 or U^2. In $\mathbb{C}^3$, every bounded homogeneous domain is again symmetric and is biholomorphic to one of the following:

(i) B_3, (ii) $B_2 \times U$, (iii) U^3,

or the future light cone

(iv) $\{(z_1, z_2, z_3) : y_3 > \sqrt{y_1^2 + y_2^2}\}$.

In $\mathbb{C}^n$, the number of nonequivalent bounded symmetric domains is finite. E. Cartan [Ca] proved that there exist only six types of irreducible bounded symmetric domains, the so called four classical domains and two exceptional domains of dimensions 16 and 27 respectively. The book by L. K. Hua ([Hu]) is an excellent introduction to harmonic analysis on the classical Cartan domains. Closely related to the bounded symmetric domains are the generalized half-planes or Siegel domains of type II ([Gi, Ko1]). Every bounded symmetric domain has a realization of this type.

Let Ω be a domain in $\mathbb{C}^n$. A differential operator D on $C^\infty(\Omega)$ is said to be invariant if

$$D(f \circ \psi) = (D\,f) \circ \psi \qquad \text{for all } \psi \in Aut(\Omega).$$

It is well known ([He1, He2]) that if Ω is a symmetric domain, then the algebra of invariant differential operators on $C^\infty(\Omega)$ is finitely generated. It is also known that the Laplace-Beltrami operator $\widetilde{\Delta}_\Omega$ with respect to the Bergman metric on a bounded domain Ω is invariant.

The two common global definitions of harmonic functions on a bounded domain in $\mathbb{C}^n$ are as follows:

(4) A C^2 function $f : \Omega \to \mathbb{R}$ is **invariant harmonic** or **weakly harmonic** if $\widetilde{\Delta}_\Omega f = 0$, where $\widetilde{\Delta}_\Omega$ is the Laplace-Beltrami operator on Ω.

(5) A C^∞ function $f : \Omega \to \mathbb{R}$ is **strongly harmonic** if $Df = 0$ for all invariant differential operators D on $C^\infty(\Omega)$ for which $D\,1 = 0$.

Since the Laplace-Beltrami operator is invariant, we clearly have (5) $\subset$ (4), and except in the case of the ball, the containment is proper. For symmetric domains, or symmetric spaces in general, there is an abundance of information about strongly harmonic functions. A nice survey of the subject matter may be found in [Ko3, Ko4]. In 1963, Furstenberg [Fu1] (see also [Fu2]) using probability theory and Lie group machinery obtained a Poisson integral representation for bounded strongly harmonic functions on Riemannian symmetric spaces of noncompact type, which includes the bounded symmetric domains. In the same paper he also proved that in this setting every bounded weakly harmonic function is strongly harmonic.

For arbitrary domains, there are very few results concerning weakly harmonic and subharmonic functions. In the notes we will develop many of the basic properties of invariant (or weakly) harmonic and subharmonic functions on the unit ball of $\mathbb{C}^n$. In the remarks we will point out some of the results which are known in other settings such as the polydisc, or bounded symmetric domains. In many instances however, the solutions of the analogous problems, even for simple domains such as the polydisc, or the pseudoconvex ellipsoidal domains

$$D_\alpha = \{(z_1, z_2) : |z_1|^{2/\alpha} + |z_2|^2 < 1\}, \quad \alpha \neq 1,$$

are in most cases unknown.

For the polydisc U^n, there is a significant difference between strongly harmonic and weakly harmonic functions. As we will see in Sections 3.2 and 4.4, a function on U^n is strongly harmonic (subharmonic) if and only if it is n-harmonic (n-subharmonic). In Section 3.2, we show that for U^n, the Laplace-Beltrami operator $\widetilde{\Delta}_{U^n}$ is given by

$$\widetilde{\Delta}_{U^n} = 2\sum_{j=1}^{n}(1-|z_j|^2)^2 \frac{\partial^2}{\partial z_j \partial \overline{z}_j},$$

and we also give an example of a function f which is weakly harmonic, but not strongly harmonic. Even though strongly harmonic and subharmonic functions on the polydisc have been studied since the late 1930's, the few results concerning weakly harmonic functions are much more recent. Fatou's theorem for weakly harmonic functions on U^n was not proved until 1983 (see Section 7.6), and at present, the appropriate analogue of Littlewood's theorem for weakly subharmonic functions appears to be unknown.

As was indicated above, for the unit ball B, the concepts of weakly harmonic and strongly harmonic functions coincide. In Section 3.3, we will show that the Laplace-Beltrami operator on B is given by

$$\widetilde{\Delta}_B = \frac{4}{(n+1)}(1-|z|^2)\sum_{i,j=1}^{n}[\delta_{i,j} - \overline{z}_i z_j]\frac{\partial^2}{\partial z_j \partial \overline{z}_i}.$$

Functions on B which are harmonic with respect to this operator are ususally referred to as **invariant** harmonic or $\mathcal{M}$-**harmonic**. As we will see in Section 4.1, this is equivalent to f satisfying the non-euclidean mean value property

$$f(\psi(0)) = \int_S f(\psi(rt))\, d\sigma(t), \qquad 0 < r < 1,$$

for all $\psi \in Aut(B)$.

There are some significant differences between the usual Laplacian Δ as given in (1), and the Laplace-Beltrami operator $\widetilde{\Delta}_B$. The ordinary Laplacian Δ is uniformly elliptic on B. This is not the case for the operator $\widetilde{\Delta}$, which is degenerate on the boundary. Since Δ is uniformly elliptic on B, if f is C^∞ on the boundary S of B, and if F is the solution to the Dirichlet problem for Δ, then F is C^∞ on $\overline{B}$. For the operator $\widetilde{\Delta}$ this result fails dramatically. In

Section 5.2 we give an example of a C^∞ function on S for which the solution F of the Dirichlet problem for $\widetilde{\Delta}$ is not C^2 on S.

Another significant difference is as follows: for the usual Laplacian, $\Delta f_r = r^2 \Delta f$, where for $0 < r < 1$, $f_r(z) = f(rz)$. Thus if f is harmonic or subharmonic, the same is true for f_r. This fails for the operator $\widetilde{\Delta}$, and as a consequence, many of the classical results about harmonic or subharmonic functions which relied on contractions, require new proofs. Even standard results, such as proving that a bounded harmonic function can be represented as the Poisson integral of a bounded function, become nontrivial for $\mathcal{M}$-harmonic functions on the ball. It is partly for this reason that the Poisson integral formula for bounded strongly harmonic functions remained unsolved until the work of Furstenberg in 1963.

In Sections 5.1 and 5.2 we introduce the Poisson-Szegö kernel and consider the Dirichlet problem for $\widetilde{\Delta}$ on the ball. These results seem to have appeared first in 1958 in the book by L. K. Hua [Hu]. In Section 5.3 we prove the Poisson integral formula for functions in the Hardy space $\mathcal{H}^p$, $1 \leq p \leq \infty$, of $\mathcal{M}$-harmonic functions on B. The Poisson integral formula itself follows from a more general result about $\mathcal{M}$-subharmonic functions. In Theorem 5.8, using an equicontinuity argument due to Ullrich, we prove that if f is a continuous $\mathcal{M}$-subharmonic function on B satisfying

$$\sup_{0<r<1} \int_S |f(rt)|^p \, d\sigma(t) < \infty, \qquad 1 \leq p < \infty,$$

or is bounded when $p = \infty$, then f has an $\mathcal{M}$-harmonic majorant which is the Poisson-Szegö integral of an L^p function when $p > 1$, and a measure when $p = 1$. The argument given also proves the Poisson integral formula for functions in the Hardy space $\mathcal{H}^p$, $1 \leq p \leq \infty$.

In Chapter 6 we derive the Green's function for $\widetilde{\Delta}$ and prove the Riesz decomposition theorem for $\mathcal{M}$-subharmonic functions having an $\mathcal{M}$-harmonic majorant. As an application of the results of this chapter, we prove that a harmonic function f is in $\mathcal{H}^p$, $1 < p < \infty$, if and only if

$$\int_B (1 - |z|^2)^n |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 \, d\lambda(z),$$

where $\widetilde{\nabla}$ denotes the invariant gradiant on B, and λ is the invariant volume measure on B. The above also characterizes the classical Hardy spaces H^p of holomorphic functions on B for all p, $0 < p < \infty$. When $n = 1$, $|\widetilde{\nabla} f(z)|^2 = 2(1 - |z|^2)^2 |f'(z)|^2$, for holomorphic functions f.

Chapters 7 and 8 are devoted to extensions to the ball of Fatou's theorem and Littlewood's theorem concerning boundary limits of $\mathcal{M}$-harmonic and

$\mathcal{M}$-subharmonic functions on B. For Fatou's theorem, the significant difference between the euclidean and non-euclidean case is that the appropriate analogue of the non-tangential approach regions are the admissible regions of Koranyi. For a point $\zeta \in S$, these are non-tangential along the complex line $\{\lambda\zeta : \lambda \in \mathbb{C}, |\lambda| < 1\}$, but are tangential in the orthogonal direction.

In Chapter 8, in addition to proving Littlewood's theorem concerning radial limits of $\mathcal{M}$-subharmonic functions, we also prove the existence of admissible limits for $\mathcal{M}$-subharmonic functions f for which the Riesz measure $\widetilde{\Delta} f$ is absolutely continuous and satisfies

$$\int_B (1-|w|^2)^n (\widetilde{\Delta} f(w))^p \, d\lambda(w) < \infty,$$

for some $p > n$. Section 8.3 also contains recent results on tangential limits of invariant Green potentials in B. In Chapter 9 we present several results concerning invariant Riesz potentials and L^p estimates for the invariant gradient of Green potentials of functions on B. These are motivated by well known results about the classical Riesz kernel on $\mathbb{R}^n$.

In the final chapter we include some recent results on weighted Bergman and Dirichlet type spaces of $\mathcal{M}$-harmonic functions on B. Section 10.1 contains several useful mean value inequalities for both $|h|^p$ and $|\widetilde{\nabla} h|^p$, $0 < p < \infty$, for $\mathcal{M}$-harmonic functions h on B. These inequalities are used in the final two sections to first prove a generalization of a theorem of Hardy and Littlewood comparing the rate of growth of the p'th means of h and $\widetilde{\nabla} h$, and in the study of weighted Bergman and Dirichlet type spaces on B. For $0 < p < \infty$, $\gamma \in \mathbb{R}$, the weighted Dirichlet space $\mathcal{D}_p^\gamma$ is defined as the space of $\mathcal{M}$-harmonic functions h on B for which

$$\int_B (1-|z|^2)^\gamma |\widetilde{\nabla} h(z)|^p \, d\lambda(z) < \infty.$$

Similarly, the weighted Bergman space $\mathcal{A}_p^\gamma$ consists of those $\mathcal{M}$-harmonic functions h for which

$$\int_B (1-|z|^2)^\gamma |h(z)|^p \, d\lambda(z) < \infty.$$

One of the main results of Section 10.3 is that for $p \geq 1$, $\gamma > n$, $h \in \mathcal{A}_p^\gamma$ if and only if $h \in \mathcal{D}_p^\gamma$.

1.
Notation and Preliminary Results

In this chapter we introduce some basic notation and some preliminary results which will be used throughout the notes. The notation is as in [Ru3], and many of the proofs of some of the preliminary results can be found in that text.

1.1. Notation.

As usual, $\mathbb{C}^n$ will denote n-dimensional complex space with inner product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j, \qquad (z, w \in \mathbb{C}^n),$$

and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. The standard orthonormal basis elements in $\mathbb{C}^n$ are denoted by $e_1, ..., e_n$. For $a \in \mathbb{C}^n$, $r > 0$, let

$$B(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}.$$

For simplicity, the unit ball $B(0,1)$ will be denoted by either B or B_n. The boundary of B is the unit sphere $S = \{z : |z| = 1\}$. When $n = 1$, the unit disc in $\mathbb{C}$ will be denoted by U, and for $n > 1$,

$$U^n = \{z \in \mathbb{C}^n : \Delta(z) < 1\},$$

where

$$\Delta(z) = \max\{|z_j| : j = 1, 2, ..., n\}.$$

The set $T^n = \{z : |z_j| = 1,\ j = 1, 2, ..., n\}$ is called the distinguished boundary of U^n.

Since our discussion of functions of several complex variables requires multi-index notation, we introduce the following standard conventions. Let $N = \{0, 1, 2, ...\}$ denote the set of natural numbers. A multi-index α is an ordered n-tuple

$$\alpha = (\alpha_1, ..., \alpha_n) \qquad \text{with } \ \alpha_j \in N,\ j = 1, ..., n.$$

For a multi-index α and $z \in \mathbb{C}^n$, set

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_n, \\ \alpha! &= \alpha_1! \cdots \alpha_n!, \\ z^\alpha &= z_1^{\alpha_1} \cdots z_n^{\alpha_n}. \end{aligned}$$

If Ω is an open subset of $\mathbb{C}^n$, $k \in N$, we denote by $C^k(\Omega)$ the space of real or complex valued function on Ω which have continuous derivatives of order α for all multi-indices α with $|\alpha| \le k$. $C_c^k(\Omega)$ denotes those functions in $C^k(\Omega)$ with compact support. The meanings of $C^\infty(\Omega)$ and $C_c^\infty(\Omega)$ are as usual.

Let Ω be an open subset of $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is **holomorphic** if f is holomorphic in each variable separately, i.e., for each $a \in \Omega$ and each $i = 1, ..., n$, the function

$$\lambda \to f(a + \lambda e_i)$$

is holomorphic in an open neighborhood of 0 in $\mathbb{C}$. It is an old result due to Hartogs that if f is holomorphic as defined above, then f is continuous in Ω, and as a consequence of the Cauchy integral formula for polydiscs, for every $a \in \Omega$, f has a power series expansion

$$f(z) = \sum_\alpha a_\alpha (z-a)^\alpha,$$

which converges absolutely in a neighborhood $B(a, r)$ of a, and uniformly on compact subsets of $B(a, r)$, where for $\alpha = (\alpha_1, ..., \alpha_n)$,

$$a_\alpha = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(a).$$

As in the case $n = 1$, if $z_j = x_j + iy_j$, with x_j and y_j real, the partial differential operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ are defined by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

If f is holomorphic, it is an immediate consequence of the definition that

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \quad \text{for all} \quad j = 1, ..., n.$$

For a C^1 function f, this condition is also sufficient that f be holomorphic.

If $w = \varphi(z) = (\varphi_1(z), ..., \varphi_k(z))$ is a C^1 mapping of a domain $\Omega_1 \subset \mathbb{C}^n$ into a domain $\Omega_2 \subset \mathbb{C}^k$, and f is a C^1 function on Ω_2, then for $g(z) = f(\varphi(z))$, the following complex versions of the chain rule hold:

$$\frac{\partial g}{\partial z_j} = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial w_i} \frac{\partial w_i}{\partial z_j} + \frac{\partial f}{\partial \overline{w}_i} \frac{\partial \overline{w}_i}{\partial z_j} \right) \tag{1.1}$$

and

$$\frac{\partial g}{\partial \overline{z}_j} = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial w_i} \frac{\partial w_i}{\partial \overline{z}_j} + \frac{\partial f}{\partial \overline{w}_i} \frac{\partial \overline{w}_i}{\partial \overline{z}_j} \right) \tag{1.2}$$

If φ is a holomorphic mapping, i.e., $\varphi_j(z)$ is holomorphic for all $j = 1, ..., k$, then $\dfrac{\partial w_i}{\partial \overline{z}_j} = 0$ for all i, j. Thus if f is holomorphic, $f \circ \varphi$ is also holomorphic. If Ω_1, Ω_2 are domains in $\mathbb{C}^n$, a one-to-one holomorphic mapping φ of Ω_1 onto Ω_2 with holomorphic inverse is called a **biholomorphic mapping** of Ω_1 onto Ω_2. That φ is one-to-one and onto is sufficient that the inverse map is holomorphic and that $\det J_\varphi(z) \neq 0$ for all $z \in \Omega_1$, where J_φ is the Jacobian matrix of the mapping φ given by

$$J_\varphi(z) = \left(\frac{\partial \varphi_i(z)}{\partial z_j} \right)_{i,j=1}^{n}. \tag{1.3}$$

For a domain $\Omega \subset \mathbb{C}^n$, the group (under composition) of all biholomorphic mappings of Ω onto Ω is called the automorphism group of Ω and is denoted by $Aut(\Omega)$.

Let Ω be a domain in $\mathbb{C}^n$. An upper semicontinuous function $f : \Omega \to [-\infty, \infty)$ with $f \not\equiv -\infty$ is **plurisubharmonic (p.s.h.)** on Ω if for each $a \in \Omega$, $w \in \mathbb{C}^n$, the function

$$\lambda \to f(a + \lambda w) \tag{1.4}$$

is subharmonic in a neighborhood of 0 in $\mathbb{C}$. A continuous function $f : \Omega \to \mathbb{R}$ is **pluriharmonic** if the function (1.4) is harmonic in a neighborhood of 0 in $\mathbb{C}$ for every $a \in \Omega$ and $w \in \mathbb{C}^n$.

For a C^2 function f, if one computes the Laplacian of the function (1.4) at 0 by using the complex form of the chain rules (1.1) and (1.2), one obtains that f is p.s.h. on Ω if and only if

$$\sum_{i,j=1}^{n} \frac{\partial^2 f(z)}{\partial z_i \partial \overline{z}_j} w_i \overline{w}_j \geq 0 \tag{1.5}$$

for all $z \in \Omega$ and $w \in \mathbb{C}^n$.

1.2. Integral Formulas on B.

Let ν denote Lebesgue measure in $\mathbb{C}^n$ normalized so that $\nu(B) = 1$. If V denotes the usual Euclidean volume measure in $\mathbb{C}^n$ and $c_n = V(B_n)$, then $c_n d\nu = dV$. Also, let σ denote the rotation-invariant measure on S also normalized so that $\sigma(S) = 1$. The term rotation-invariant refers to the orthogonal group $O(2n)$ of all isometries of $\mathbb{R}^{2n}$ which fix the origin.

The following integration formulas, the proofs of which may be found in [Ru3], will prove useful throughout.

$$\int_{\mathbb{C}^n} f\,d\nu = 2n\int_0^\infty r^{2n-1}\int_S f(r\zeta)\,d\sigma(\zeta)\,dr. \tag{1.6}$$

$$\int_S f\,d\sigma = \int_S \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}\zeta)\,d\theta\,d\sigma(\zeta). \tag{1.7}$$

$$\int_S f\,d\sigma = \int_{B_{n-1}} \frac{1}{2\pi}\int_0^{2\pi} f(\zeta', e^{i\theta}\zeta_n)\,d\theta\,d\nu(\zeta'). \tag{1.8}$$

$$\int_S f\,d\sigma = \int_{\mathcal{U}} f(U\eta)\,dU. \tag{1.9}$$

In (1.8), $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$, and in (1.9), dU denotes the Haar measure on the group $\mathcal{U} = \mathcal{U}(n)$ of unitary transformations of $\mathbb{C}^n$. A linear transformation $U \in \mathcal{U}$ if and only if

$$\langle Uz, Uw\rangle = \langle z, w\rangle$$

for all $z, w \in \mathbb{C}^n$. The group $\mathcal{U}$ is a compact subgroup of $O(2n)$. In addition to the above, if f is a function of one complex variable only, then for $n > 1$, $\eta \in S$,

$$\int_S f(\langle\zeta,\eta\rangle)\,d\sigma(\zeta) = \frac{n-1}{\pi}\iint_U (1-r^2)^{n-2} f(re^{i\theta})\,r\,dr\,d\theta. \tag{1.10}$$

Since we will have occassion to use both the binomial and multinomial expansion formulas, we include them at this point. For $|\lambda| < 1$, β not a negative integer,

$$(1-\lambda)^{-\beta} = \sum_{k=0}^\infty \frac{\Gamma(k+\beta)}{\Gamma(\beta)\,k!}\lambda^k, \tag{1.11}$$

where Γ denotes the Gamma function. Also, for $k = 1, 2, \ldots$,

$$\langle z, w\rangle^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \overline{w}^\alpha. \tag{1.12}$$