1 Exterior Algebra

Anyone who has studied linear algebra and vector calculus may have wondered whether the notion of cross product of vectors in 3-dimensional space generalizes to higher dimensions. Exterior algebra, which is a prerequisite for the study of differential forms, shows that the answer is yes. We shall adopt a constructive approach to exterior algebra, following closely the presentation given in Flanders [1989], and we will try to emphasize the connection with the vector algebra notions of cross product and triple product (see Table 1.2 on page 19).

1.1 Exterior Powers of a Vector Space

1.1.1 The Second Exterior Power

Let V be an n-dimensional vector space over R. Elements of V will be denoted u, v, w, u_i, v^j, etc., and real numbers will be denoted a, b, c, a_i, b_j, etc. For $p = 0, 1, \ldots, n$, the pth exterior power of V, denoted $\Lambda^p V$, is a real vector space, whose elements are referred to as "***p*-vectors**." For $p = 0, 1$ the definition is straightforward: $\Lambda^0 V = R$, and $\Lambda^1 V = V$, respectively. $\Lambda^2 V$, consists of formal sums[1]

$$\sum_i a_i (u_i \wedge v_i), \tag{1.1}$$

where the "wedge product" $u \wedge v$ satisfies the following four rules:

$$(au+w) \wedge v = a(u \wedge v) + w \wedge v; \tag{1.2}$$

[1] A rigorous construction of the second exterior power is given in Section 1.9.

$$u \wedge (bv + w) = b(u \wedge v) + u \wedge w; \tag{1.3}$$

$$u \wedge u = 0; \tag{1.4}$$

$$\{v^1, \ldots, v^n\} \text{ is a basis for } V \Rightarrow \{v^i \wedge v^j : 1 \le i < j \le n\} \text{ is a basis for } \Lambda^2 V. \tag{1.5}$$

Postponing to the end of this chapter the question of whether a vector space with these properties exists, let us note two immediate consequences of (1. 2), (1. 3), and (1. 4). Apply (1. 4) to $(u + v) \wedge (u + v)$, and then express the latter as the sum of four terms using (1. 2) and (1. 3); two of these terms, namely, $u \wedge u$ and $v \wedge v$, are zero, and what remains shows that $u \wedge v + v \wedge u = 0$; hence

$$v \wedge u = -u \wedge v. \tag{1.6}$$

Second (1. 2), (1. 3), and (1. 4) by themselves imply that, for any basis $\{v^1, \ldots, v^n\}$ of V, the set of vectors $\{v^i \wedge v^j : 1 \le i < j \le n\}$ spans $\Lambda^2 V$, because it spans the set of "generators" $\{u \wedge w, u \text{ and } w \in V\}$; to check this, we express u and w in terms of the basis $\{v^1, \ldots, v^n\}$, and apply (1. 2), (1. 3), and (1. 6) to obtain:

$$u \wedge w = \left(\sum a_i v^i\right) \wedge \left(\sum b_j v^j\right) = \sum_{i,j} a_i b_j (v^i \wedge v^j)$$

$$= \sum_{i<j} (a_i b_j - a_j b_i)(v^i \wedge v^j).$$

The linear independence of $\{v^i \wedge v^j : 1 \le i < j \le n\}$ cannot, however, be deduced from (1. 2), (1. 3), and (1. 4), and is studied in Section 1.9.

1.1.2 Higher Exterior Powers

The description of $\Lambda^p V$ for any $2 \le p \le n$ follows the same lines; $\Lambda^p V$ is the set of formal sums[2]

$$\sum_{\gamma} a_{\gamma} (u_{\gamma(1)} \wedge \ldots \wedge u_{\gamma(p)}) \tag{1.7}$$

of "generators" $u_{\gamma(1)} \wedge \ldots \wedge u_{\gamma(p)}$, where each coefficient a_{γ} is indexed by a multi-index $\gamma = (\gamma(1), \ldots, \gamma(p))$; elements of $\Lambda^p V$ are called "***p*-vectors**," and are subject to the rules (1. 8), (1. 9), and (1. 10):

$$(av + w) \wedge u_2 \wedge \ldots \wedge u_p = a(v \wedge u_2 \wedge \ldots \wedge u_p) + w \wedge u_2 \wedge \ldots \wedge u_p, \tag{1.8}$$

[2] A rigorous construction is given in Section 1.9.

and similarly if any of the u_i is replaced by such a linear combination;

$$u_i = u_j \text{ for some } i \neq j \Rightarrow u_1 \wedge \ldots \wedge u_p = 0; \tag{1.9}$$

and for any basis $\{v^1, \ldots, v^n\}$ of V, the following set of p-vectors forms a basis for $\Lambda^p V$:

$$\{v^{i(1)} \wedge \ldots \wedge v^{i(p)}, 1 \leq i(1) < \ldots < i(p) \leq n\} \tag{1.10}$$

The expression $u_1 \wedge \ldots \wedge u_{r-1} \wedge (v+w) \wedge u_{r+1} \wedge \ldots \wedge (v+w) \wedge \ldots \wedge u_p$, which is zero by (1. 9), can be expanded using (1. 8) into four terms, two of which are zero; what remains shows that

$$u_1 \wedge \ldots \wedge u_p \text{ changes sign if any two entries are transposed.} \tag{1.11}$$

Also it follows from (1. 8) and (1. 9) that, for any basis $\{v^1, \ldots, v^n\}$ of V, the set of vectors (1. 10) spans $\Lambda^p V$; in order to demonstrate this, we shall need the language of permutations.

1.1.3 Permutations

Let Σ_p denote the set of permutations of the set $\{1, 2, \ldots, p\}$. For example, Σ_3 can be written as $\{e, (1,2), (3,1), (2,3), (1,3,2), (1,2,3)\}$, where $\pi = (3,1)$ means for example that $\pi(1) = 3, \pi(3) = 1$. A **transposition** is an element π of Σ_p that switches i and j for some $i \neq j$, but leaves k fixed for all $k \notin \{i, j\}$; thus in the list for Σ_3 above, the second, third, and fourth elements are transpositions. A result in algebra states that any permutation can be expressed as a composition of transpositions, and that the number m of transpositions is unique modulo 2; we define the **signature** $\mathrm{sgn}(\pi)$ of the permutation π by

$$\mathrm{sgn}(\pi) = (-1)^m. \tag{1.12}$$

It is also true, in the case of the composition $\pi \bullet \pi'$ of two permutations, that $\mathrm{sgn}(\pi \bullet \pi') = \mathrm{sgn}(\pi)\,\mathrm{sgn}(\pi')$. It follows from (1. 11) that

$$u_{\pi(1)} \wedge \ldots \wedge u_{\pi(p)} = \mathrm{sgn}(\pi)\,(u_1 \wedge \ldots \wedge u_p). \tag{1.13}$$

Now we will show how to express an arbitrary generator of $\Lambda^p V$ as a linear combination of the set of vectors (1. 10). We may write

$$u_1 \wedge \ldots \wedge u_p = \Big(\sum_{j(1)} b_{1,j(1)} v^{j(1)}\Big) \wedge \ldots \wedge \Big(\sum_{j(p)} b_{p,j(p)} v^{j(p)}\Big)$$

$$= \sum_J c_J (v^{j(1)} \wedge \ldots \wedge v^{j(p)}),$$

where $J = (j(1), \ldots, j(p))$, and $c_J = b_{1,j(1)} \ldots b_{p,j(p)}$. For any J, there is a unique multi-index $I = (i(1), \ldots, i(p))$ such that $i(1) < \ldots < i(p)$, and a unique $\pi \in \Sigma_p$ such that $J = \pi(I)$, meaning that $(j(1), \ldots, j(p)) = (\pi(i(1)), \ldots, \pi(i(p)))$. Hence by (1. 13), we deduce

$$v^{j(1)} \wedge \ldots \wedge v^{j(p)} = \operatorname{sgn}(\pi) \, (v^{i(1)} \wedge \ldots \wedge v^{i(p)}),$$

and therefore

$$u_1 \wedge \ldots \wedge u_p = \sum_I \Big(\sum_\pi \operatorname{sgn}(\pi) \, c_{\pi(I)}\Big) (v^{i(1)} \wedge \ldots \wedge v^{i(p)}), \qquad (1.\ 14)$$

where the first summation is over multi-indices I such that $i(1) < \ldots < i(p)$, and the second summation is over Σ_p. This completes the proof that the vectors (1. 10) span $\Lambda^p V$.

1.1.4 Calculating the Dimension of an Exterior Power

$$\dim(\Lambda^p V) = \frac{n!}{(n-p)!\,p!}, \; 0 \le p \le n. \qquad (1.\ 15)$$

Proof: For any basis $\{v^1, \ldots, v^n\}$ of V, the set of p-vectors

$$\{v^{i(1)} \wedge \ldots \wedge v^{i(p)}, 1 \le i(1) < \ldots < i(p) \le n\}. \qquad (1.\ 16)$$

forms a basis for $\Lambda^p V$, by (1. 10). The number of elements of this set is the number of ways of choosing p objects from n distinct objects, which is the expression shown. ¤

Let us illustrate these ideas by writing down bases for the exterior powers of R^3.

p	Basis for $\Lambda^p V$	Dimension
0	$\{1\}$	1
1	$\{e_1, e_2, e_3\}$	3
2	$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$	3
3	$\{e_1 \wedge e_2 \wedge e_3\}$	1

Table 1.1 Exterior powers of Euclidean 3-space

1.2 Multilinear Alternating Maps and Exterior Products

For any set V, the set-theoretic product $V \times \ldots \times V$ (p copies) simply means the set of ordered p-tuples $(u_1, \ldots, u_p)$ where each $u_i \in V$. If V and W are vector spaces, a mapping $h: V \times \ldots \times V \to W$ is called:

- **Multilinear** if $h(au + bu', u_2, \ldots, u_p) = ah(u, u_2, \ldots, u_p) + bh(u', u_2, \ldots, u_p)$, and similarly for the other $(p-1)$ entries of h; h is called **bilinear** if p is 2;
- **Antisymmetric** (or **alternating**) if

$$h(u_{\pi(1)}, \ldots, u_{\pi(p)}) = \operatorname{sgn}(\pi)\, h(u_1, \ldots, u_p), \pi \in \Sigma_p, \tag{1.17}$$

which implies $h(u_1, \ldots, u_p) = 0$ if $u_i = u_j$, some $i \neq j$; for when $u_i = u_j$, some $i \neq j$, transposing the ith and jth entries shows that $h(u_1, \ldots, u_p)$ is the same as its negative.

The student will have encountered the following examples of multilinear alternating maps in linear algebra or vector calculus courses:

$$(u,v) \to u \times v, R^3 \times R^3 \to R^3;$$

$$(u,v) \to \left|\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|, R^2 \times R^2 \to R;$$

$$(u, v, w) \to u \cdot (v \times w), R^3 \times R^3 \times R^3 \to R.$$

The linear maps from V to W will be denoted $L(V \to W)$, and the multilinear alternating maps will be denoted $A_p(V \to W)$. The following property of exterior powers will play a central role in the remainder of this chapter.

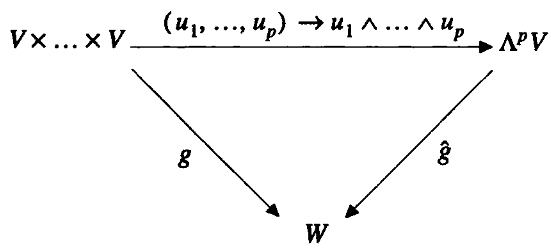
1.2.1 Universal Alternating Mapping Property

To every $g \in A_p(V \to W)$, there corresponds a unique $\hat{g} \in L(\Lambda^p V \to W)$ such that

$$\hat{g}(u_1 \wedge \ldots \wedge u_p) = g(u_1, \ldots, u_p), \forall u_1, \ldots, u_p; \tag{1.18}$$

in other words, a unique $\hat{g}$ such that the following diagram commutes.[3]

[3] A diagram is said to commute if following any sequence of arrows from one set to another yields the same mapping.



Proof: Deferred to Section 1.9.

1.2.2 Exterior Products

There exists a unique bilinear map $(\lambda,\mu) \to \lambda \wedge \mu$ from $\Lambda^p V \times \Lambda^q V$ to $\Lambda^{p+q} V$, whose effect on generators is that

$$(u_1 \wedge \ldots \wedge u_p) \wedge (w_1 \wedge \ldots \wedge w_q) = u_1 \wedge \ldots \wedge u_p \wedge w_1 \wedge \ldots \wedge w_q. \tag{1.19}$$

To see that this is true, apply 1.2.1 twice: first to the multilinear, alternating map

$$(u_1, \ldots, u_p) \to u_1 \wedge \ldots \wedge u_p \wedge w_1 \wedge \ldots \wedge w_q,$$

for fixed $w_1 \wedge \ldots \wedge w_q$, so as to obtain a unique $f \in L(\Lambda^p V \to \Lambda^{p+q} V)$ such that

$$f(u_1 \wedge \ldots \wedge u_p) = u_1 \wedge \ldots \wedge u_p \wedge w_1 \wedge \ldots \wedge w_q,$$

so that we may define

$$\lambda \wedge (w_1 \wedge \ldots \wedge w_q) = f(\lambda);$$

and second to the multilinear, alternating map

$$(w_1, \ldots, w_q) \to \lambda \wedge (w_1 \wedge \ldots \wedge w_q),$$

for fixed λ, so as to obtain $g_\lambda \in L(\Lambda^q V \to \Lambda^{p+q} V)$ such that

$$g_\lambda (w_1 \wedge \ldots \wedge w_q) = \lambda \wedge (w_1 \wedge \ldots \wedge w_q). \tag{1.20}$$

Finally the **exterior product** of $\lambda \in \Lambda^p V$ and $\mu \in \Lambda^q V$ is defined by $\lambda \wedge \mu = g_\lambda(\mu)$. The properties of the exterior product, the first two of which are immediate from the preceding construction, are:

- $(\lambda,\mu) \to \lambda \wedge \mu$ is **distributive** over addition and scalar multiplication;
- **associativity**: $(\lambda \wedge \mu) \wedge \nu = \lambda \wedge (\mu \wedge \nu)$;
- $\mu \wedge \lambda = (-1)^{pq} (\lambda \wedge \mu)$, so two vectors of odd degrees **anticommute**; otherwise the vectors commute.

The last property follows from Exercise 4 below, in the case where λ, μ are generators, and in general from linearity. In order to obtain a practical grasp of exterior products, try Exercises 5 and 6 below.

1.2.3 Example

Suppose V is 4-dimensional with a basis $\{v^1, v^2, v^3, v^4\}$. Then

$$(a(v^3 \wedge v^4) + b(v^1 \wedge v^3)) \wedge (c(v^1 \wedge v^2) + d(v^1 \wedge v^4)) = ac(v^3 \wedge v^4 \wedge v^1 \wedge v^2)$$
$$= (-1)^{2(2)} ac(v^1 \wedge v^2 \wedge v^3 \wedge v^4).$$

1.3 Exercises

1. (a) Repeat Table 1.1 for the case of R^4, using the basis $\{e_1, e_2, e_3, e_4\}$.
(b) Let $u = ae_1 + ce_3$, $v = be_2 + de_4$; express $u \wedge v$ in terms of your basis of $\Lambda^2 R^4$.
(c) Let $w = a'e_1 + b'e_2$; express $u \wedge v \wedge w$ in terms of your basis of $\Lambda^3 R^4$.
(d) Express $u \wedge v \wedge w \wedge e_3$ in terms of your basis of $\Lambda^4 R^4$.

2. Verify that, when $V = R^3$, the cross product $(u,v) \to u \times v$, $R^3 \times R^3 \to R^3$, and the triple product $(u, v, w) \to u \cdot (v \times w)$, $R^3 \times R^3 \times R^3 \to R$, are multilinear, alternating maps.[4]

Reminder: The cross product of $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ is

$$u \times v = \left|\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}\right| e_1 - \left|\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}\right| e_2 + \left|\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right| e_3, \tag{1.21}$$

and the triple product satisfies

$$u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v) = -(w \cdot (v \times u)). \tag{1.22}$$

3. Decompose the permutation $(6, 4, 3, 2, 1, 5) \in \Sigma_6$ into a product of transpositions in two different ways, and show that the number of transpositions used is the same modulo 2 in both cases.

4. Prove, by induction or otherwise, that a permutation which sends $(1, 2, \ldots, p + q)$ into $(q + 1, \ldots, q + p, 1, 2, \ldots, q)$ has signature

[4] By the end of this chapter, the reader will realize that, in terms of the star operator discussed in Section 1.7 below, $u \times v = *(u \wedge v)$, and $u \cdot (v \times w) = *(u \wedge v \wedge w)$.

$$\operatorname{sgn}(\pi) = (-1)^{pq}. \tag{1.23}$$

Hint: A possible inductive hypothesis H_k is that whenever $p \geq 1, q \geq 1, p+q \leq k$, then the assertion above holds. To prove H_{k+1} from H_k, start by transposing q and $q+p$, and then rearrange $(q, 1, \ldots, q-1)$ so that H_k can be applied to the first $p+q-1$ entries.

5. Let $V = R^3$, with any basis $\{v^1, v^2, v^3\}$; show that

$$(a(v^2 \wedge v^3) + b(v^3 \wedge v^1) + c(v^1 \wedge v^2)) \wedge (\tilde{a}v^1 + \tilde{b}v^2 + \tilde{c}v^3)$$

$$= (a\tilde{a} + b\tilde{b} + c\tilde{c})\, v^1 \wedge v^2 \wedge v^3. \tag{1.24}$$

6. Suppose V is 4-dimensional with a basis $\{v^1, v^2, v^3, v^4\}$. Express the following as multiples of $v^1 \wedge v^2 \wedge v^3 \wedge v^4$:
(i) $(a(v^1 \wedge v^3) + b(v^2 \wedge v^4)) \wedge (c(v^1 \wedge v^3) + d(v^2 \wedge v^4))$;
(ii) $(av^1 + bv^4) \wedge (c(v^1 \wedge v^2 \wedge v^3) + d(v^2 \wedge v^3 \wedge v^4))$.

7. The setting is the same as for Exercise 6. Suppose $\mu \in \Lambda^3 V, \mu \neq 0$. Characterize the vectors $u \in V$ such that $u \wedge \mu = 0$, and show that the vector space consisting of such u is of dimension 3.

Hint: Write $u = u_1 v^1 + \ldots + u_4 v^4$, and express μ similarly in terms of the four basis elements of the third exterior power. Obtain a linear relation on the coefficients of u.

8. This is a generalization of Exercise 7. Suppose V is n-dimensional, and μ is an arbitrary nonzero element of $\Lambda^{n-1} V$. Prove that the subspace W^{μ} of elements u of V such that $u \wedge \mu = 0$ is of dimension $n-1$, and deduce from this that there exist vectors $w^1, \ldots, w^{n-1}$ in V such that $\mu = w^1 \wedge \ldots \wedge w^{n-1}$.

Hint: For the last part, take a basis for W^{μ}, extend it to a basis for V, and express μ in terms of the corresponding basis of $\Lambda^{n-1} V$. **Warning:** This kind of representation does not generally hold for elements of the other exterior powers.

1.4 Exterior Powers of a Linear Transformation

1.4.1 Determinants

Given $A \in L(V \to V)$, define $g_A : V^n \to \Lambda^n V \cong R$ by

$$g_A(u_1, \ldots, u_n) = (Au_1) \wedge \ldots \wedge (Au_n). \tag{1.25}$$

It follows immediately from the last equation that g_A is multilinear and antisymmetric, and so, by 1.2.1, there is a unique $f_A \in L(\Lambda^n V \to \Lambda^n V)$ such that

$$f_A(u_1 \wedge \ldots \wedge u_n) = (Au_1) \wedge \ldots \wedge (Au_n). \tag{1.26}$$

Since $\Lambda^n V$ is one-dimensional and f is linear, it follows that f is simply multiplication by a scalar, which we denote by $|A|$, the **determinant** of A. In other words,

$$|A|\,(u_1 \wedge \ldots \wedge u_n) = (Au_1) \wedge \ldots \wedge (Au_n). \tag{1.27}$$

It is somewhat surprising to discover that this abstract formulation refers to the same notion of determinant that the student has encountered in matrix algebra:

1.4.2 Formula for the Determinant of a Matrix

Suppose that, in terms of a basis $\{v^1, \ldots, v^n\}$ for V, A has the matrix representation $A = (a_{ij})_{1 \le i,j \le n}$ (a_{ij} may also be written $a_{i,j}$). Then taking

$$u_i = \sum_j a_{ij} v^j$$

gives, as in (1. 14),

$$u_1 \wedge \ldots \wedge u_n = \Big(\sum_{j(1)} a_{1,j(1)} v^{j(1)}\Big) \wedge \ldots \wedge \Big(\sum_{j(n)} a_{n,j(n)} v^{j(n)}\Big),$$

$$= \sum_J a_{1,j(1)} \cdots a_{n,j(n)} (v^{j(1)} \wedge \ldots \wedge v^{j(n)}),$$

where $J = (j(1), \ldots, j(n))$. Any J with two entries the same makes no contribution to the sum, by (1. 9). In all other cases there is a unique $\pi \in \Sigma_n$ such that $(j(1), \ldots, j(n)) = (\pi(1), \ldots, \pi(n))$. Hence by (1. 13), we deduce

$$v^{j(1)} \wedge \ldots \wedge v^{j(n)} = \operatorname{sgn}(\pi)\,(v^1 \wedge \ldots \wedge v^n),$$

$$u_1 \wedge \ldots \wedge u_n = \left(\sum_{\pi \in \Sigma_n} \operatorname{sgn}(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}\right) v^1 \wedge \ldots \wedge v^n. \tag{1.28}$$

Thus the formula for the determinant of the matrix is

$$|A| = \sum_{\pi \in \Sigma_n} \operatorname{sgn}(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}. \tag{1.29}$$

For example, when $n = 2$,

$$\left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = a_{11}a_{22} - a_{12}a_{21} = \sum_{\pi \in \Sigma_2} \operatorname{sgn}(\pi)\, a_{1,\pi(1)} a_{2,\pi(2)}.$$

1.4.3 Other Exterior Powers of a Linear Transformation

A generalization of the notion of determinant is that of **exterior powers**[5] **of a linear transformation** $A \in L(V \to W)$. The map $V^p \to \Lambda^p W$ given by

$$(u_1, \ldots, u_p) \to (Au_1) \wedge \ldots \wedge (Au_p)$$

is multilinear and alternating, and so by 1.2.1 it defines an element of $L(\Lambda^p V \to \Lambda^p W)$ denoted $\Lambda^p A$, called the **exterior** ***p*****th power** of A; in other words, $\Lambda^p A$ is specified by its action on generators as follows:

$$\Lambda^p A(u_1 \wedge \ldots \wedge u_p) = (Au_1) \wedge \ldots \wedge (Au_p). \tag{1.30}$$

The matrix representation of $\Lambda^p A$ may be obtained as follows. If $\{v^1, \ldots, v^n\}$ is a basis for V, and $\{w^1, \ldots, w^m\}$ for W, then $\{\sigma^I\}$ and $\{\tau^K\}$ are bases for $\Lambda^p V$ and $\Lambda^p W$, respectively, where

$$\sigma^I = v^{i(1)} \wedge \ldots \wedge v^{i(p)},\ 1 \le i(1) < \ldots < i(p) \le n; \tag{1.31}$$

$$\tau^K = w^{k(1)} \wedge \ldots \wedge w^{k(p)},\ 1 \le k(1) < \ldots < k(p) \le m. \tag{1.32}$$

If $Av^i = \sum_k a^i_k w^k$, then

$$(\Lambda^p A)\sigma^I = (Av^{i(1)}) \wedge \ldots \wedge (Av^{i(p)})$$

$$= \sum_J a^{i(1)}_{j(1)} \ldots a^{i(p)}_{j(p)} (w^{j(1)} \wedge \ldots \wedge w^{j(p)}),$$

where J runs through the set of all multi-indices. As usual, summands where $j(r) = j(s)$ for some $r \neq s$ are zero, and we express the other summands as in the steps preceding (1. 14): there is a unique $K = (k(1), \ldots, k(p))$ such that $k(1) < \ldots < k(p)$, and a unique $\pi \in \Sigma_p$ such that $J = \pi(K)$, meaning that $(j(1), \ldots, j(p)) = (\pi(k(1)), \ldots, \pi(k(p)))$. Since

$$w^{j(1)} \wedge \ldots \wedge w^{j(p)} = \operatorname{sgn}(\pi)\,(w^{k(1)} \wedge \ldots \wedge w^{k(p)}),$$

we obtain:

[5] This idea is needed in calculations related to the pullback of differential forms in Chapter 2, and is also relevant to Stokes's Theorem in Chapter 8.