Representations of Groups on Finite Simplicial Complexes

MICHAEL ASCHBACHER

California Institute of Technology

Until recently there have been relatively few articles in the finite group theoretic literature on representations of finite groups on simplicial complexes. However in the last few years that situation has begun to change. This paper discusses some of the activity in the area. We begin with a fairly general discussion intended to give a feeling for the kind of activity now going on. Since space is limited, we touch on only a few examples of such activity, and to provide focus, we eventually concentrate on p-group complexes of finite groups. In the end we concentrate even further on a particular problem in the area of p-group complexes: the question of when p-group complexes of finite groups are simply connected. There we go into more detail.

This volume is devoted to groups of Lie type and their geometries. The p-group complexes of a group G should be viewed as geometries for G. The p-group complexes of the groups of Lie type will be featured prominently here. In particular we will see that if G is of Lie type and characteristic p then the p-group complexes of G are homotopy equivalent to the building of G.

The term "simplicial complex" is used here to mean an abstract simplicial complex. Thus a *simplicial complex* K consists of a set K of objects called *vertices* together with a collection of finite subsets of K called *simplices* such that each subset of a simplex is a simplex. The term simplicial complex is often used in the topological literature for a geometric realization of the abstract complex; the reader may be more familiar with this latter usage.

A simplex $s = \{x_0, \ldots, x_k\}$ is of *dimension* k if it has $k+1$ vertices. The *dimension* of K is the maximum dimension of a simplex of K. Most of the complexes we will consider are finite. Morphisms in the category are the *simplicial maps* which are maps of vertices which take simplices to simplices.

Examples (1) Let Δ be a graph. The *clique complex* $K(\Delta)$ of Δ is the simplicial complex whose vertices are the vertices of Δ and whose simplices are the finite cliques of Δ. Conversely given a simplicial complex L the *graph* $\Delta(L)$ of L is the graph on the vertices of L with x adjacent to y if

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$\{x, y\}$ is a simplex. Notice L is a subcomplex of $K(\Delta(L))$. L is said to be *connected* if its graph is connected.

(2) Let G be a finite group, p a prime, and $\Lambda_p(G)$ the commuting graph on the set of all subgroups of G of order p. The *commuting complex* $K_p(G)$ of G is the clique complex $K(\Lambda_p(G))$. The commuting complex is one of the p-group complexes of G. Those with some background in finite simple group theory know that the graph $\Lambda_p(G)$ has long been important in simple group theory, that it is elementary and well known that $\Lambda_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup, and that groups with a strongly p-embedded subgroup are of great importance in the proof of the Classification of the finite simple groups.

(3) Define a *geometric complex* to be a Tits geometry Γ on an index set I together with a set $\mathcal{C}$ of chambers of Γ (*ie.* flags of type I) such that each flag of rank at most 2 is contained in some member of $\mathcal{C}$. The complex can be regarded as a simplicial complex whose vertices are the objects of Γ and simplices are the nonempty flags contained in members of $\mathcal{C}$. Thus the geometric complex is a subcomplex of the clique complex of Γ. Those familiar with chamber systems will observe that the category of geometric complexes is isomorphic to the category of chamber systems X such that for each $x \in X$ and index $j \in I$, $\{x\} = \bigcap_i [x]_{i'}$ and $[x]_{j'} = \bigcap_{i \in j'} [x]_{i'}$. Chamber systems were introduced by Tits in part because the category of Tits geometries is too small. I find the simplicial complex point of view a more geometric and intuitive way to extend the category of geometries.

(4) Let G be a group, and $\mathcal{F} = (G_i : i \in I)$ a finite family of subgroups of G. Define $\mathcal{C}(G, \mathcal{F})$ to be the geometric complex whose vertex set is the union of the coset spaces G/G_i, $i \in I$, with a set s of vertices a simplex if and only if $\bigcap_{X \in s} X \neq \varnothing$. Call such a complex a *coset complex*.

In this example each vertex $G_i x$ has a type $\tau(G_i x) = i \in I$ and the maximal simplices are indeed of type I; *eg.* if s is a simplex then there is $x \in \bigcap_{X \in S} X$ and $s = S_{J,x} = \{G_j x : j \in J\}$, where $J = \tau(s)$, so $s = S_{J,x} \subseteq S_{I,x}$ of type I. Finally G is represented as a group of automorphisms of $\mathcal{C}(G, \mathcal{F})$ via right multiplication and G is transitive on simplices of type J for each $J \subseteq I$. Conversely any geometric complex K on I admitting a group of automorphisms transitive on simplices of each type is isomorphic to a coset complex.

(5) Let P be a poset. The *order complex* $\mathcal{O}(P)$ of P is the simplicial complex whose vertices are the members of P and whose simplices are the finite chains in P. We often write P for $\mathcal{O}(P)$.

(6) Let G be a finite group, p a prime, and $\mathcal{S}_p(G)$ the set of all nontrivial p-subgroups of G partially ordered by inclusion. We also write $\mathcal{S}_p(G)$ for the order complex of this poset, and call this complex the *Brown complex* of G at p. The subcomplex $\mathcal{A}_p(G)$ of all elementary abelian p-subgroups is

the *Quillen complex.* Thus we have two more p-group complexes associated to G. There are still others.

A *geometric realization* of K is simply an identification of the vertices of K with suitable points in some Euclidean space, with a simplex identified with the convex closure of its vertices. The realization is then regarded as a topological subspace of Euclidean space. We can then define two complexes to be *homotopy equivalent* if their geometric realizations are homotopy equivalent.

Invariants of the homotopy type of the complex are its homology groups $H_n(K)$ and its fundamental group $\pi_1(K)$. These can be defined combinatorially without reference to the geometric realization.

Examples (7) K is *contractible* if it has the same homotopy type as a point. In that case the reduced homology of K and its fundamental group are trivial. Recall K is *acyclic* if its reduced homology is trivial and K is *simply connected* if its fundamental group is trivial. So contractible complexes are acyclic and simply connected.

(8) Let G be a finite group and p a prime. The commuting complex, the Brown complex, and the Quillen complex at the prime p are all of the same homotopy type.

(9) Let K be a simplicial complex. The *barycentric subdivision* $sd(K)$ of K is the order complex of the poset of simplices of K ordered by inclusion. It is well known that $sd(K)$ has the same homotopy type as K.

One of the most useful tools available in this area is to replace a complex by another complex of the same homotopy type which may be better suited to analysis. We will see this tool used many times. Here are three lemmas which allow us to make such replacements:

LEMMA 1. *Let $f : P \to Q$ be a map of posets. Then*

*(1) (Quillen [**Q**]) If $f^{-1}(Q(\geq q))$ is contractible for all $q \in Q$ then $f : \mathcal{O}(P) \to \mathcal{O}(Q)$ is a homotopy equivalence.*

*(2) [**A**] Assume for each $q \in Q$, $f^{-1}(Q(\geq q))$ is $min\{1, h(q) - 1\}$-connected, $Q(> q)$ is connected if $h(q) = 0$, and if $h(q) = 1$ then either $Q(> q) \neq \varnothing$ or $f^{-1}(Q(\geq q))$ is simply connected. Then P is simply connected if Q is simply connected.*

Here $-1, 0, 1$-connectivity means nonempty, connected, simply connected, respectively. $Q(\geq q) = \{x \in Q : x \geq q\}$ and $Q(> q)$, $Q(\leq x)$, and $Q(< x)$ are defined similarly. The *height* of $q \in Q$ is $h(q) = dim(Q(\leq q))$.

LEMMA 2. *Let C be a cover of a simplicial complex K by subcomplexes. The nerve $N(C)$ of C is the complex with vertex set C and $s \subseteq C$ a simplex if and only if $I(s) = \bigcap_{X \in C} X \neq \emptyset$. Assume $I(s)$ is empty or contractible for all s. Then $N(C)$ and K have the same homotopy type.*

LEMMA 3. *(Quillen [**Q**]) Let G be a group of automorphisms of the poset P and let Q be the order complex of some poset of subgroups of G. Assume*

(1) For each $x \in P$, $Q \cap G_x$ is contractible (simply connected).

(2) For each $X \in Q$, $Fix_P(X)$ is contractible (simply connected).

(3) If $X \in Q$ and $x \in Fix_P(X)$ then $P(\leq x) \subseteq Fix_P(X)$.

Then P and Q have the same homotopy type. (P is simply connected if and only if Q is simply connected.)

There are two directions one can go in the area. First, use techniques and ideas from combinatorial topology to prove things about finite groups and geometries. Second, use modern finite group theory and finite geometry to prove results in combinatorial topology. We give a quick example of each approach.

Recall K is simply connected if $\pi_1(K)$ is trivial. Equivalently K has no proper connected coverings. Coverings of K correspond to topological coverings of its geometric realization. Combinatorially a covering of K is a surjective local isomorphism $f : \tilde{K} \to K$ of simplicial complexes.

In simple group theory we often wish to characterize a group G as the unique group satisfying suitable hypotheses. In particular in the Classification of the finite simple groups we need to characterize each simple group via suitable hypotheses on local subgroups, preferably centralizers of involutions. In [**AS1**], Yoav Segev and I came up with an approach to this problem with a topological flavor. In brief, given some hypotheses $\mathcal{H}$ on the centralizer of an involution in some finite simple group G, we produce a family $\mathcal{F}$ of subgroups of G and form the coset complex $K = \mathcal{C}(G, \mathcal{F})$. For $J \subset I$ let $G_J = \bigcap_{j \in J} G_j$, be the stablizer of the simplex $S_{J,1}$ of type J. The inclusion maps $G_J \to G_K$ for $K \subset J$ define an *amalgam* $\mathcal{A}$ of groups and there is a largest group $\tilde{G}$ realizing this family and a surjective local isomorphism $\alpha : \tilde{G} \to G$. We show $\mathcal{A}$ is determined up to isomorphism; hence if $\bar{G}$ is any group satisfying hypotheses $\mathcal{H}$, there is a local isomorphism $\bar{\alpha} : \tilde{G} \to \bar{G}$.

Now $\tilde{G}$ acts on its coset complex $\tilde{K}$ and there is a covering $f : \tilde{K} \to K$ of complexes. Our approach is to show K is simply connected; hence f and therefore also α is an isomorphism. But then $\bar{\alpha} : G \to \bar{G}$ is a group homomorphism and as G is simple, $\bar{\alpha}$ is an isomorphism, so G is determined up to isomorphism.

Segev and I usually use a lot of knowledge of the complex K and some simple minded techniques to show K is simply connected. It would be good to have more powerful techniques.

So far the applications to finite group theory involve only low dimensional properties of complexes; *ie.* connectivity and simple connectivity are properties of the 1-skeleton and 2-skeleton. However Alperin's Conjecture can be stated in terms of the p-group complexes; this may turn out to be an application which uses the full strength of the theory. Also Webb has results which describe the p-part of the cohomology of a finite group G in terms of the cohomology of stabilizers of simplices in any p-group complex of G, perhaps making possible an inductive approach to the cohomology of finite groups. See Webb's survey article [**W**] for a discussion of these topics.

Let us next turn to an example of how finite simple group theory can be used to answer a question from combinatorial topology. Let K be a finite acyclic simplicial complex and $G \leq Aut(K)$. Replacing K by its barycentric subdivision, we can assume with little loss of generality that the action of G on K is *admissible*; that is if $g \in G$ fixes a simplex then it fixes each vertex of the simplex. The Lefschetz Fixed Point Theorem says that g has a fixed point on K; but what about G? Constructions of Robert Oliver [**O**] show that for most finite groups G there exist finite acyclic complexes K such that G has no fixed points on K. But if the dimension of K is small one can hope to prove something.

CONJECTURE. *(Warren Dicks) If G is a finite group acting admissibly on a 2-dimensional contractible complex then G has a fixed point.*

The Conjecture fails if we weaken "contractible" to acyclic; the Poincaré dodecahedron disk is a 2-dimensional acyclic complex admitting the fixed point free action of A_5. Its fundamental group is $SL_2(5)$.

Tensoring the simplicial chain complex with any ring R we get homology $H_*(K,R)$ with coeficients in R. K is R-acyclic if $\tilde{H}_*(K,R) = 0$. If π is a set of primes define K to be π-acyclic if K is F- acyclic for each field F of characteristic p and each $p \in \pi$.

THEOREM. *(Aschbacher-Segev [**AS3**]) If G is finite group and K is $\pi(G)$-acyclic finite complex with G admissible on K then either G has a fixed point on K or G has a composition factor which is a rank 1 group of Lie type or J_1.*

We also construct a family of 2-dimensional complexes admitting the action of $G = PGL_2(q)$, which appear to be $\pi(G)$-acyclic when q is even. We call these complexes *polygon complexes*. Each is of the form $\mathcal{C}(G,\mathcal{F})$, where $\mathcal{F} = (G_1, G_2, G_3)$ with G_1 a Borel subgroup of G and G_2 and G_3 normalizers of suitable representatives of the 2 classes of maximal tori. More concretely, one can think of the coset space G/G_1 as points on the projective line $X = GF(q) \cup \{\infty\}$, G/G_2 as edges $\{x,y\} \in X \times X$, and G/G_3 as

$q+1$-gons on X. The Poincaré dodecahedron disk is the polygon complex for $q = 4$. Using Mathematica we calculated the homology groups for $q \leq 16$ and Richard Wilson made a pseudo calculation for $q = 32$ for one of the complexes. The complexes were p-acyclic for all primes distinct from 17,1087,239,$\{67, 659, 1033\}$, for $q = 8$, 16,16,32, respectively. We find these primes mysterious.

This leaves open the Dicks Conjecture. One can also ask if A_5 is the unique simple group acting without fixed points on a 2-dimensional $\mathbf{Z}$-acyclic complex.

We now concentrate on p-subgroup complexes of finite groups. The Brown and Quillen complexes of a finite group G were introduced by Brown [**B**] and Quillen [**Q**], presumably to study group cohomology. As indicated earlier, the graph of the commuting complex has long been important in simple group theory. To get some feel for these complexes let us consider an example:

Example (10) Let G be a finite simple group of Lie type in characteristic p. Then the building of G is a finite simplicial complex; indeed the building is the coset complex of the family of maximal parabolics over a fixed Borel subgroup. It is easy to show that the building has the same homotopy type as $\mathcal{S}_p(G)$. By the Solomon-Tits Theorem, the building is *spherical*; that is it is simply connected if the dimension is at least 2, and reduced homology vanishes except in the top dimension. The top dimensional homology is the Steinberg module for G.

More generally we would like to know the homology and fundamental group of $\mathcal{S}_p(G)$, for G simple and p a prime divisor of G, and we want results which reduce problems about p-group complexes of general finite groups to problems about simple groups. In a moment, we illustrate with a discussion of simply connectivity of p-group complexes.

The problem on p-group complexes with the most history and visibility is the Quillen Conjecture:

Quillen Conjecture. *Let G be a finite group and p a prime. Then $\mathcal{S}_p(G)$ is contractible if and only if $O_p(G) \neq 1$.*

One direction is easy: if $O_p(G) \neq 1$ then $\mathcal{S}_p(G)$ is contractible. The opposite direction was proved by Quillen [**Q**] for solvable groups. It is known to be true for simple groups. In [**ASm**], Steve Smith and I show that if $p > 5$ and G has no components which are unitary groups then the Conjecture holds. If one could show the top dimensional homology of $\mathcal{A}_p(U_n(q))$ is nontrivial for p dividing $q+1$ then the Conjecture would (essentially) be settled for $p > 5$.

In the remainder of the talk we will concentrate on the following question:

QUESTION 1. *Let G be a finite group and p a prime. When is $K_p(G)$ simply connected?*

If $O_p(G) \neq 1$ we just saw that $K_p(G)$ is contractible and hence simply connected, so the interesting case occurs when $O_p(G) = 1$. We will see that in that event a necessary condition for $K_p(G)$ to be simply connected is that $m_p(G) > 2$. (Recall $m_p(G)$ is the *p-rank* of G; *ie.* the largest n such that G has a subgroup isomorphic to n copies of the group of order p.) The answer to Question 1 seems to be that if $m_p(G) > 2$ then $K_p(G)$ is almost always simply connected. As is usual in finite group theory, a strategy for proving this conjecture involves two steps:

(1) Reduction: Reduce the simple connectivity of $K_p(G)$ for the general finite group G to the simply connectivity of $K_p(G_0)$ for certain minimal groups G_0.
(2) Analysis of Minimal Groups: Determine when $K_p(G_0)$ is simply connected for the minimal groups G_0.

The reduction step has been accomplished in [**A**]. Minimal groups include the simple groups. Analysis of the minimal groups is begun in [**A**], [**S**], and [**D**]. I regard this project as a test case for a more ambitious analysis of the behavior of the p-group complexes of finite groups; *eg.* obtaining a qualitative description of the homology of these complexes including a reduction theory, and a precise description of the homology of an appropriate set of minimal groups.

Our first set of minimal groups is the set Min_0 of groups G such that $G = HA$ with A an elementary abelian p-group, H a normal subgroup of G of order prime to p, $O_p(G) = 1$, and G is minimal subject to these constraints. The case where G is solvable was handled by Quillen in [**Q**]:

THEOREM. *(Quillen) If G is solvable in Min_0 then $\mathcal{A}_p(G)$ is spherical. That is $\tilde{H}_k(\mathcal{A}_p(G)) = 0$ for $k < dim(\mathcal{A}_p(G)) = m_p(G) - 1$ and $\mathcal{A}_p(G)$ is simply connected if $m_p(G) > 2$.*

One can ask if $G_0 \in Min_0$ is nonsolvable is Quillin's result still true? We conjecture at least:

CONJECTURE. *Let G be a finite group such that $G = AF^*(G)$, where A is an elementary abelian p-subgroup of rank at least 3 and $F^*(G)$ is the direct product of the A-conjugates of a simple component L of G of order prime to p. Then $K_p(G)$ is simply connected.*

The following result from [A] supplies strong evidence for the conjecture, in essence reducing it to the case where L is a group of Lie type and Lie rank 1 or a sporadic group.

THEOREM 3. *Assume G and L satisfy the hypotheses of the Conjecture and that the Conjecture holds in proper sections of G. Then*

(1) If L is of Lie type and Lie rank at least 2 then $K_p(G)$ is simply connected.

(2) If $L \cong L_2(q)$ with q even then $K_p(G)$ is simply connected.

(3) If L is an alternating group then $K_p(G)$ is simply connected.

(4) If L is a Mathieu group then $K_p(G)$ is simply connected.

The minimal groups not in Min_0 are the finite simple groups of p-rank at least 3. The next two theorems from [A] supply the reduction step which reduces the question of simple connectivity of $K_p(G)$ to the case G minimal. Given a graph Λ and a vertex x of Λ, write $\Lambda(x)$ for the set of vertices distinct from x and adjacent to x in Λ.

THEOREM 1. *Assume the Conjecture and let G be a finite group, p a prime divisor of the order of G, and $\Lambda = \Lambda_p(G)$. Assume $\Lambda(x)$ is connected for all $x \in \Lambda$ and let $\bar{G} = G/O_{p'}(G)$. Then exactly one of the following holds:*

(1) $K_p(G)$ is simply connected.

(2) $\bar{G} = \bar{G}_1 \times \bar{G}_2$ and $\bar{G}_i$ has a strongly p-embedded subgroup for $i = 1$ and 2.

(3) $\bar{G} = \bar{X}(\bar{G}_1 \times \bar{G}_2)$, for some $X \in \Lambda$, $p = 3, 5$, $\bar{G}_1 \cong L_2(8)$, $Sz(32)$, respectively, $\bar{G}_2$ is a nonabelian simple group with a strongly p-embedded subgroup, and X induces outer automorphisms on $\bar{G}_i$ for $i = 1$ and 2.

(4) $\bar{G}$ is almost simple and $K_p(\bar{G})$ and $K_p(F^*(\bar{G}))$ are not simply connected.

THEOREM 2. *Let G be a finite group, p a prime divisor of the order of G, and assume $O_p(G) = 1$, $\Lambda = \Lambda_p(G)$ is connected, and $H_1(K_p(G)) = 0$. Then $m_p(G) > 2$ and $\Lambda(x)$ is connected for each $x \in \Lambda_p(G)$.*

Theorems 1 and 2 say that, modulo the Conjecture and a short list of exceptions, $K_p(G)$ is simply connected if and only if $m_p(G) > 2$ and $\Lambda(x)$ is connected for each $x \in \Lambda_p(G)$.

The following observations expand upon these points:

Remarks (1) If $O_p(G) \neq 1$ then G is contractible and hence simply connected. Thus the restriction that $O_p(G) = 1$ in Theorem 2 causes no loss of generality.

(2) It is well known that $\Lambda_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup. (*cf.* 44.6 in [**FGT**]) Moreover we know all groups with strongly p-embedded subgroups. (*cf.* [**A**]) Thus the restriction in Theorem 2 that Λ be connected results in no loss of generality, and the groups in Cases (2) and (3) of Theorem 1 are completely described.

(3) Recall a simplicial complex is simply connected if and only if its fundamental group is trivial, while the first homology group of the complex is the abelianization π_1/π_1' of its fundamental group. Thus the hypothesis in Theorem 2 that $H_1(K_p(G)) = 0$ is weaker than simple connectivity. So Theorem 2 says that the hypothesis in Theorem 1 that $\Lambda(x)$ be connected for each $x \in \Lambda$ is necessary for simple connectivity, and that if $O_p(G) = 1$ and $K_p(G)$ is simply connected then $m_p(G) > 2$.

(4) The condition that $\Lambda(x)$ be connected has various equivalent formulations; it is roughly equivalent to $C_G(x) = \Gamma_{2,P}(C_G(x))$ for $P \in Syl_p(C_G(x))$, for those readers who are simple group theorists. All finite groups G with $m_p(G) \geq 3$ such that $\Lambda(x)$ is disconnected for some $x \in \Lambda$ are determined in [**A**]. Thus Theorems 1 and 2 constitute a fairly complete reduction to the simple case, modulo the Conjecture.

(5) Recall from the Classification of the finite simple groups that each nonabelian simple group L is an alternating group, a group of Lie type, or one of the 26 sporadic groups. Thus Theorem 3 reduces a verification of the Conjecture to the case where L is of Lie type and Lie rank 1 (*ie.* $L \cong L_2(q)$, $U_3(q)$, $Sz(q)$, or ${}^2G_2(q)$) or L is one of the 21 sporadic groups which are not Mathieu groups. We will see one possible approach to handling these groups in a while.

In short Theorems 1 through 3 reduce the problem of determining when $K_p(G)$ is simply connected to (a) the problem of verifying the Conjecture for rank 1 groups of Lie type and the sporadic groups, and (b) determining when $K_p(G)$ is simply connected for G a finite simple group.

We now try to give enough of an idea of the proof of Theorem 1 to see how the minimal groups arise. Recall part 2 of Lemma 1. Let H be a normal subgroup of order prime to p, $\bar{G} = G/H$, $P = \mathcal{A}_p(G)$, and $Q = \mathcal{A}_p(\bar{G})$. Let $f : P \to Q$ be the map $f(A) = \bar{A}$. Then $f^{-1}(Q(\leq \bar{A})) = \mathcal{A}_p(AH)$, so to achieve the hypotheses of Lemma 1, part 2, we need to know $\mathcal{A}_p(AH)$ is simply connected if $h(\bar{A})) > 1$; as $h(\bar{A}) = m_p(A) - 1$, this is equivalent to $m_p(A) > 2$. This can be proved by an easy induction argument if it holds for $AH \in Min_0$.

We also need to know that $Q(> \bar{A})$ is connected if $h(\bar{A}) = 0$; *ie.* if A is of order p then $m_p(C_G(A)) > 1$, which follows if $m_p(G) > 1$. Finally if $h(\bar{A}) = 1$ we need $Q(> \bar{A}) \neq \varnothing$; *ie.* if $m_p(A) = 2$ then $m_p(C_G(A)) > 2$, which holds if $m_p(G) > 2$ and $\Lambda(x)$ is connected for each x of order p in A.

So Lemma 1 allows us to conclude:

LEMMA 4. *Assume the conjecture with $m_p(G) > 2$, let H be a normal p'-subgroup of G, $\Lambda(x)$ connected for all x of order p in G, and $\mathcal{A}_p(G/H)$ simply connected. Then $\mathcal{A}_p(G)$ is simply connected.*

The converse of Lemma 4 is fairly easy to prove, so we can reduce to the case $O_{p'}(G) = O_p(G) = 1$. To reduce to the case G simple requires:

LEMMA 5. *Let $H \trianglelefteq G$ such that $K_p(C_H(x))$ is connected for all $x \in \Lambda(G) - \Lambda(H)$ and $K_p(H)$ is simply connected. Then $K_p(G)$ is simply connected.*

Lemma 5 is proved using results of Yoav Segev and the author on simple connectivity of simplicial complexes found in [**AS2**].

Let us next consider the Conjecture. The following lemma is a result of Segev in [**S**] extending a result in [**A**] used to prove Theorem 3:

LEMMA 6. *Assume the hypotheses of the conjecture and let $B = N_A(L)$. Assume $\mathcal{F}$ is a family of B-invariant proper subgroups of L such that*

(1) $N_L(X) \cap C_L(B) \leq X$ for each $X \in \mathcal{F}$.
(2) $\mathcal{C}(G, \mathcal{F})$ is simply connected.
(3) $\mathcal{A}_p(\langle X, A\rangle)$ is simply connected for each $X \in \mathcal{F}$.
(4) $link_{\mathcal{C}(G,\mathcal{F})}(X)$ is connected for all $X \in \mathcal{X}$.
(5) The truncation of $\mathcal{C}(G, \mathcal{F})$ at any 2-subset of $\mathcal{F}$ is connected.
(6) If $B \neq 1$ then $C_L(B) = \langle C_X(B) : X \in \mathcal{F}\rangle$.
Then G satisfies the Conjecture.

Hypothesis (3) holds automatically in a minimal counter example to the conjecture. Hypotheses (4) and (5) hold if the coset complex $\mathcal{C}(G, \mathcal{F})$ is residually connected. If A is regular on the components of G then (6) is vacuously satisfied; if not L is of Lie type and B is of order p inducing field automorphisms on L. The critical condition is the hypothesis that the coset complex be simply connected. That is why small groups like groups of Lie type of Lie rank 1 cause problems; their subgroup structure is not rich enough to have a coset complex for which simple connectivity is easy to verify. However Segev has shown in [**S**] that:

THEOREM 4. *(Segev) Assume the hypotheses of the Conjecture with A regular on the components of G and $L \cong L_2(q)$, $Sz(q)$, or $U_3(r)$, $r \equiv 0, 1$ mod 3. Then the Conjecture holds.*