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The Algebraic Characterization of Geometric 4-Manifolds

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Contents

Preface

It is well known that every closed surface admits a geometry of constant curvature, and that such surfaces may be classified up to homeomorphism either by their fundamental group or by their Euler characteristic and orientation character. Much current research in dimension 3 is guided by the expectation that all closed 3-manifolds have decompositions into geometric pieces, and that (lens spaces aside) the homeomorphism type is essentially determined by the fundamental group. (Here the Euler characteristic is always 0).

In dimension 4 there is no reason to expect that every closed 4-manifold may have a geometric decomposition, and the Euler characteristic and fundamental group are independent invariants. Nevertheless the closed 4-manifolds which admit geometries or fibre over a geometric base with geometric fibre form a large and interesting class. In these notes we shall attempt to characterize algebraically such 4-manifolds (up to homotopy equivalence or homeomorphism). This task has three main parts: finding complete invariants for the homotopy type, determining which systems of invariants are realizable and applying surgery (where possible) to obtain s-cobordisms. In many cases the Euler characteristic, fundamental group and Stiefel-Whitney classes together form a complete system of invariants for the manifold and the possible invariants can be described explicitly. We shall also see that such bundle spaces usually have the minimal Euler characteristic for their fundamental groups. (The only exceptions have base and fibre S^2 or RP^2).

Our results are most satisfactory for infrasolvmanifolds. We show that a closed 4-manifold M is homeomorphic to an infrasolvmanifold if and only if $\chi(M) = 0$ and $\pi_1(M)$ has a locally nilpotent normal subgroup of Hirsch length at least 3, and two such manifolds are homeomorphic if and only if their fundamental groups are isomorphic. Moreover $\pi_1(M)$ is then a torsion free virtually poly-Z group of Hirsch length 4 and every such group is the fundamental group of an infrasolvmanifold. We also consider in detail the question of when such a manifold is the mapping torus of a self homeomorphism of a 3-dimensional infrasolvmanifold.

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We also show that a closed 4-manifold M is simple homotopy equivalent to the total space of a F-bundle over B (where B and F are closed surfaces, B is aspherical and F is aspherical or S^2) if and only if $\chi(M) = \chi(B)\chi(F)$ and $\pi_1(M)$ is an extension of $\pi_1(B)$ by a normal subgroup isomorphic to $\pi_1(F)$. Any such extension is the fundamental group of such a bundle space; the bundle is determined by the group in the aspherical cases and by the group and Stiefel-Whitney classes if the fibre is S^2. The total spaces of orientable S^2-bundles over aspherical orientable surfaces are determined up to s-cobordism by theses invariants; in general there are only finitely many s-cobordism classes of such manifolds homotopy equivalent to a given bundle space.

For aspherical geometric 4-manifolds other than infrasolvmanifolds there is as yet no good intrinsic characterization of the groups arising as fundamental groups. Moreover in some other cases (notably the nonorientable total spaces of surface bundles over RP^2) we do not yet have a complete system of invariants for the homotopy type. In a recent book H.J.Baues has developed the notion of "quadratic crossed complex" and used it to show how in principle all 4-dimensional homotopy types may be classified. It should be possible to apply his ideas in these cases, but computations using quadratic crossed complexes have only been carried through for a handful of examples so far.

The major difficulty in extending this work to a classification up to homeomorphism of all such 4-manifolds is that we do not know whether s-cobordisms between 4-manifolds are always topologically products. This is only known to be so when the fundamental group is elementary amenable. Even under the latter assumption we do not know the Whitehead groups or surgery obstruction groups in most cases when the fundamental group has torsion.

The organization of this book is as follows. The first chapter is purely algebraic; here we develop the notions of elementary amenable group and safe extension of a group ring and criteria for the vanishing of cohomology of a group with coefficients in a free module. The next three chapters are homotopy theoretic. Chapter II gives general criteria for two closed 4-manifolds to be homotopy equivalent, Chapter III considers criteria for a closed 4-manifold to be homotopy equivalent to the total space of a bundle with base or fibre a circle and Chapter IV considers surface bundles. Whitehead groups and the surgery exact sequence are discussed in Chapter V. Chapters VI-IX consider the different 4-dimensional geometries, grouped according to whether the model is homeomorphic to R^4, $S^2 \times R^2$, $S^3 \times R$ or is compact. In the final chapter these results are applied to determine when the 4-manifold obtained by surgery on a 2-knot admits a geometry or a complex analytic structure, or is simple homotopy equivalent to such a manifold, and to give characterizations of minimal properly elliptic surfaces and ruled surfaces (fibred or ruled over curves of genus > 1). There is an appendix summarizing relevant properties of PD_3-complexes and a short list of open questions before the references.

I would like to thank Tom Farrell for his advice on Lemmas IV.2 and V.2, Ian Hambleton for his advice on Theorem IV.13 and Warren Dicks and Mike Mihalik for their advice on ends. I would also like to thank Michael Farber, Cherry Kearton and Jerry Levine and the Departments of Mathematics at Tel-Aviv University, the University of Durham and Brandeis University (respectively) for their hospitality and support while parts of the work presented here were done. (The balance was done at Macquarie University and the University of Sydney under the traditional conditions of academic life - that is, without external funding).

The text was prepared using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

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