CHAPTER I

ALGEBRAIC PRELIMINARIES

The key algebraic idea used in this book is to study the homology of covering spaces as modules over the group ring of the group of covering transformations. In this chapter we shall summarize the relevant notions from group theory: elementary amenable groups, finiteness conditions, the stable invariant basis number property, and the connection between ends and the vanishing of cohomology with coefficients in a free module.

Our principal references for group theory are [Bi], [DD] and [Ro].

1. Group theoretic notation and terminology

Let G be a group. Then G' and ζG denote the commutator subgroup and centre of G, respectively. The *outer automorphism group* of G is $Out(G) = Aut(G)/Inn(G)$, where $Inn(G) \cong G/\zeta G$ is the subgroup of $Aut(G)$ consiting of conjugations by elements of G. If H is a subgroup of G let $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G, respectively.

If $p : G \to Q$ is an epimorphism with kernel N we shall say that G is an *extension of* $Q = G/N$ *by the normal subgroup* N. The action of G on N by conjugation determines a homomorphism from G/N to $Out(N) = Aut(N)/Inn(N)$. If $G/N \cong Z$ the extension splits: a choice of element t in G which projects to a generator of G/N determines a right inverse to p. Let θ be the automorphism of N determined by conjugation by t in G. Then G is isomorphic to the semidirect product $N \times_\theta Z$. Every automorphism of N arises in this way, and automorphisms whose images in $Out(N)$ are conjugate determine isomorphic semidirect products. In particular, if θ is an inner automorphism then $G \cong N \times Z$.

If P and Q are classes of groups let PQ denote the class of ("P *by* Q") groups G which have a normal subgroup H in P such that the quotient G/H

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX is in Q, and let ℓP denote the class of ("*locally-P*") groups such that each finitely generated subgroup is contained in some P-subgroup. In particular, if F is the class of finite groups ℓF is the class of *locally finite* groups. Let *poly-P* be the class of groups with a finite composition series such that each subquotient is in P. Thus if Ab is the class of abelian groups poly-Ab is the class of solvable groups.

A group *virtually* has some property if it has a normal subgroup of finite index with that property. Let vP be the class of groups which are virtually P. Thus a *virtually poly-Z* group is one which has a subgroup of finite index with a composition series whose factors are all infinite cyclic. The number of infinite cyclic factors is independent of the choice of finite index subgroup or composition series, and is called the *Hirsch length* of the group. We shall later extend the group theoretic terminology to covering spaces.

The *Hirsch-Plotkin radical* $\sqrt{G}$ is the maximal locally-nilpotent normal subgroup of G; in a virtually poly-Z group every subgroup is finitely generated, and so $\sqrt{G}$ is then the maximal nilpotent normal subgroup. If H is normal in G then $\sqrt{H}$ is normal in G also, since it is a characteristic subgroup of H, and in particular it is a subgroup of $\sqrt{G}$.

2. Elementary amenable groups and Hirsch length

The class of *elementary amenable* groups is the class of groups generated from the class of finite groups and Z by the operations of extension and increasing union. This class arose first in connection with the Banach-Tarski paradox, but is of interest here as the largest class of groups over which topological surgery techniques are known to work in dimension 4. (We shall occasionally refer to the more general notion of amenable group. See [P]).

We may construct this class as follows. Let $X_0 = 1$ and $X_1 = AbF$ be the class of finitely generated virtually abelian groups. If X_α has been defined for some ordinal α let $X_{\alpha+1} = (\ell X_\alpha)X_1$ and if X_α has been defined for all ordinals less than some limit ordinal β let $X_\beta = \cup_{\alpha<\beta} X_\alpha$. Then the class of elementary amenable groups is $EA = \cup X_\alpha$, where the union is taken over all ordinals α.

The class EA is well adapted to arguments by transfinite induction on the ordinal $\alpha(G) = \min\{\alpha | G \in X_\alpha\}$. It is closed under extension (in fact $X_\alpha X_\beta \subseteq X_{\alpha+\beta}$) and increasing union, and under the formation of sub- and quotient groups. Moreover, if κ is the first uncountable ordinal then every countable elementary amenable group is in X_κ. Hence $EA = X_{\kappa+1} = \ell X_\kappa$. Torsion groups in EA are locally finite and no group in EA has a nonabelian free subgroup. Clearly every locally-finite by virtually solvable group is elementary amenable, i.e., (ℓF)*vpoly-Ab* $\subset EA$. However there are finitely generated torsion free elementary amenable groups which are not virtually solvable.

The notion of Hirsch length (as a measure of the size of a solvable group) may be extended to elementary amenable groups. The *Hirsch length* $h(G)$ of such a group G is a nonnegative integer or ∞, defined as follows. If G is in X_1 let $h(G)$ be the rank of an abelian subgroup of finite index in G. If $h(G)$ has been defined for all G in X_α and H is in ℓX_α let $h(H) = \text{l.u.b.}\{h(F) | F \leq H, F \in X_\alpha\}$. Finally, if G is in $X_{\alpha+1}$, so has a normal subgroup H in ℓX_α with G/H in X_1, let $h(G) = h(H) + h(G/H)$. Transfinite induction on $\alpha(G)$ may be used to prove (simultaneously) that h is well defined, that if H is a subgroup of G then $h(H) \leq h(G)$, that if H is a normal subgroup then $h(G) = h(H) + h(G/H)$ and that $h(G) = \text{l.u.b.}\{h(F) | F \textit{ is a finitely generated subgroup of } G\}$ [Hi91].

Lemma 1. *Let G be a finitely generated infinite elementary amenable group. Then G has normal subgroups $K < H$ such that G/H is finite, H/K is free abelian of positive rank and the action of G/H on H/K by conjugation is effective.*

Proof. We may show that G has a normal subgroup K such that G/K is an infinite virtually abelian group, by transfinite induction on $\alpha(G)$. We may assume that G/K has no nontrivial finite normal subgroup. If H is a subgroup of G which contains K and is such that H/K is a maximal abelian normal subgroup of G/K then H and K satisfy the above conditions. //

Lemma 2. *Let $G = \cup_{n\geq 0} G_n$ be a group which the union of an increasing sequence of subgroups G_n such that each subgroup G_n has a maximal solvable normal subgroup H_n of derived length at most d and index at most M. Then G has a maximal solvable normal subgroup, of derived length at most $d+2M$*

and index at most $M!$.

Proof. Define an increasing sequence of subgroups $\bar{H}_n$ such that $H_i \leq \bar{H}_i \leq G_i$ for all $i \geq 0$ by $\bar{H}_0 = H_0$ and $\bar{H}_{j+1} = \bar{H}_j H_{j+1}$. It is easily seen by induction that for each $j \geq 0$ the subgroup $\bar{H}_j$ is a solvable subgroup of derived length at most $d+M$ and index at most M in G_j. (Note that an increasing union of solvable subgroups of derived length at most $d+M$ is solvable and of derived length at most $d+M$). Similarly, $H = \cup_{j \geq 0} H_j$ is a solvable subgroup of G which is of derived length at most $d+M$. Any finite set of coset representatives for H in G must lie in some common subgroup G_j and be in distinct cosets of H_j there. Therefore the index of H in G is at most M, and so the intersection of the conjugates of H in G is a solvable normal subgroup of index at most $M!$. Therefore G has a maximal normal solvable subgroup, S say, of index at most $M!$. Since S is an extension of $S/H \cap S$ by $H \cap S$ it has derived length at most $M + d + M = d + 2M$. //

Theorem 3. *Let G be a countable torsion free elementary amenable group. If $h(G) < \infty$ then G is virtually solvable.*

Proof. We shall show by induction on h that there are functions d and M from the set of nonnegative integers to itself such that every countable torsion free elementary amenable group of Hirsch length $h < \infty$ has a maximal solvable normal subgroup of derived length at most $d(h)$ and index at most $M(h)$. Since the only such group of Hirsch length 0 is the trivial group we may set $d(0) = 0$ and $M(0) = 1$. Suppose that the result is true for all such groups with Hirsch length at most h. If G has Hirsch length $h+1$ and is finitely generated then by Lemma 1 it has normal subgroups $K < H$ such that G/H is finite, H/K is free abelian of rank $r \geq 1$ and the action of G/H on H/K by conjugation is effective. In particular, G/H is isomorphic to a finite subgroup of $GL(r, Z)$. Since the kernel of the reduction of coefficients homomorphism from $GL(r, Z)$ to $GL(r, Z/pZ)$ is torsion free for all odd primes p it follows that G/H has order at most 3^{r^2}. As K is torsion free and elementary amenable and $k = h(K) = h_1 - r \leq h$ it has a maximal solvable normal subgroup, L say, of derived length at most $d(k)$ and index at most $M(k)$, by the hypothesis of induction. Since L is characteristic in K it is normal in G. The quotient group

G/L has a free abelian normal subgroup of index at most $[G:H]+[K:L]!$. Let $M' = \max\{3^{r^2}+M(k)!|0 \leq k \leq h, r = h+1-k\}$ and $d' = d(h)+1+M'!$. Then the maximal solvable normal subgroup of G has derived length at most d' and index at most M'. As any countable group is the union of an increasing sequence of finitely generated subgroups the general case follows from Lemma 2 on setting $d(h+1) = d'+2M'$ and $M(h+1) = M'!$. //

It can be shown that $h(G) < \infty$ if and only if G has normal subgroups $K \leq H \leq G$ such that K is locally-finite, H/K is solvable and of finite Hirsch length and G/H is finite [HL92]. A virtually solvable group of finite Hirsch length and with no nontrivial locally-finite normal subgroup must be countable, by Lemma 7.9 of [Bi].

Lemma 4. *Let G be an elementary amenable group. If $h(G) = \infty$ then for every $k > 0$ there is a subgroup H of G with $k < h(H) < \infty$.*

Proof. We shall argue by induction on $\alpha(G)$. The result is vacuously true if $\alpha(G) = 1$. Suppose that it is true for all groups in X_α and G is in ℓX_α. Since $h(G) = \text{l.u.b.}\{h(F)|F \leq G, F \in X_\alpha\}$ either there is a subgroup F of G in X_α with $h(F) = \infty$, in which case the result is true by the inductive hypothesis, or $h(G)$ is the least upper bound of a set of natural numbers and the result is true. If G is in $X_{\alpha+1}$ then it has a normal subgroup N which is in ℓX_α with quotient G/N in X_1. But then $h(N) = h(G) = \infty$ and so N has such a subgroup. //

Theorem 5. *Let G be a countable elementary amenable group of finite cohomological dimension. Then $h(G) \leq c.d.G$ and G is virtually solvable.*

Proof. Since $c.d.G < \infty$ the group G is torsion free. Let H be a subgroup of finite Hirsch length. Then H is virtually solvable and $c.d.H \leq c.d.G$ so $h(H) \leq c.d.G$. The theorem now follows from Theorem 3 and Lemma 4. //

The assumption that G be countable is unnecessary. (See [HL92]).

3. Modules and finiteness conditions

Let G be a group and R a commutative ring. If $w : G \to Z^\times = \{\pm 1\} \cong Z/2Z$ is a homomorphism then $\bar{g} = w(g)g^{-1}$ defines an anti-involution on

$R[G]$. If L is a left $R[G]$-module $\bar{L}$ shall denote the *conjugate* right $R[G]$-module with the same underlying R-module and $R[G]$-action given by $l.g = \bar{g}.l$, for all $l \in L$ and $g \in G$. (We shall also use the overbar to denote the conjugate of a right $R[G]$-module). The conjugate of a free left (right) module is a free right (left) module of the same rank.

We shall also let Z^w denote the G-module with underlying abelian group Z and G-action given by $g.n = w(g)n$ for all g in G and n in Z.

Lemma 6. [Wl65] *Let G and H be groups such that G is finitely presentable and there are homomorphisms $j : H \to G$ and $\rho : G \to H$ with $\rho j = id_H$. Then H is also finitely presentable.*

Proof. Since G is finitely presentable there is an epimorphism $p : F \to G$ from a free group $F(X)$ with a finite basis X onto G, with kernel the normal closure of a finite set of relators R. We may choose elements w_x in $F(X)$ such that $j\rho p(x) = p(w_x)$, for all x in X. Then ρ factors through the group K with presentation $< X|R, x^{-1}w_x, \forall x \in X >$, say $\rho = vu$. Now uj is clearly onto, while $vuj = \rho j = id_H$, and so v and uj are mutually inverse isomomorphisms. Therefore $H \cong K$ is finitely presentable. //

A group G is FP_n if the augmentation $Z[G]$-module Z has a projective resolution which is finitely generated in degrees $\leq n$. It is FP if it has finite cohomological dimension and is FP_n for $n = c.d.G$; it is FF if moreover Z has a finite resolution consisting of finitely generated free $Z[G]$-modules. "Finitely generated" is equivalent to FP_1, while "finitely presentable" implies FP_2. Groups which are FP_2 are also said to be *almost finitely presentable.* (It remains unkown whether FP_2 groups are finitely presentable).

If the augmentation $Q[\pi]$-module Q has a finite resolution F_* by finitely generated free modules then the alternating sum $\chi(\pi) = \Sigma(-1)^i rank(F_i)$ is independent of the resolution. (If π is the fundamental group of an aspherical finite complex K then $\chi(\pi) = \chi(K)$). This definition may be extended to groups σ which have a subgroup π of finite index with such a resolution by setting $\chi(\sigma) = \chi(\pi)/[\sigma : \pi]$. (It is not hard to see that this is well defined [Se71]).

4. The SIBN property and safe extensions of group rings

Kropholler, Linnell and Moody have shown that if H is an elementary amenable group whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup then the group ring $Z[H]$ has a classical ring of fractions which is a matrix ring over a division ring [KLM88]. That is, there is a division ring D and an embedding $i : Z[H] \to M_n(D)$ (where n is the least common multiple of the orders of the finite subgroups of H) such that the image of every nonzero divisor of $Z[G]$ is invertible in $M_n(D)$ and every element of $M_n(D)$ can be uniquely expressed in the form $i(\delta)^{-1}i(\gamma)$ for some γ and δ in $Z[H]$. Since every finitely generated free left $M_n(D)$-module is a finite dimensional left D-vector space, every onto endomorphism of such a module is an isomorphism. A ring R for whch this holds is said to have the *strong invariant basis number* (SIBN) property. Equivalently, a ring R has the SIBN property if whenever an R-module L satisfies $L \oplus R^a \cong R^b$ for some nonnegative integers a, b then $b - a$ depends only on L and is 0 if and only if $L = 0$. (Note that if a ring has the SIBN property then so does any subring).

Kaplansky showed that group rings have the SIBN property (see page 122 of [K]). Rosset extended his argument, to show that if a group G has a torsion free abelian normal subgroup A then the multiplicative system $S = C[A]\backslash\{0\}$ is an Ore system in $C[G]$, and the localization $C[G]_S$ is a safe extension of $Z[G]$ [Ro84]. We shall need a further extension of this result, due essentially to Linnell. If G is a group and $w : G \to Z/2Z$ is a homomorphism we shall say that an extension of rings $Z[G] \subseteq \Phi$ is a *safe extension* if Φ has an involution extending that of $Z[G]$, Φ has the SIBN property, Φ is flat as a right $Z[G]$=module and $\Phi \otimes_{Z[G]} Z = 0$.

Theorem 7. [Li91] *Let G be a group. Then $Z[G]$ has the SIBN property. If G has a nontrivial elementary amenable normal subgroup N whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup then $Z[G]$ has a safe extension.*

Proof. We shall only outline the argument of Kaplansky, Rosset and Linnell, refering to [Ro84] and [Li91] for further details.

Let $B = C[G]$ and equip B with the involution given by $(cg)^* = \bar{c}g^{-1}$ for

c in C and g in G, and the inner product given by $(\Sigma a_g g, \Sigma b_g g) = \Sigma a_g \bar{b}_g$. Let H be the Hilbert space completion of B. Elements of $C[N]$ act by left multiplication as bounded operators on H. The function sending ξ in $C[N]$ to the corresponding operator T_ξ embeds $C[N]$ in $B(H)$, the ring of bounded operators on H. According to [Ro68] the weak closure W of $C[N]$ in $B(H)$ can be embedded in a ring $\tilde{W}$ of densely defined operator on H which has the SIBN property and is in which every principal (left) ideal is generated by a projection which lies in W. By Theorem 4 of [Li91], if ξ is a nonzerodivisor of $C[N]$ then T_ξ is injective. Now $\tilde{W}T_\xi = \tilde{W}e$ for some idempotent e in W. Since $T_\xi(1-e) = 0$ we must have $e = 1$ and so T_ξ has a left inverse. Since $\tilde{W}$ has the SIBN property it follows that T_ξ is invertible. Thus if $F = M_n(D)$ is the classical ring of fractions for $Z[N]$ it embeds in $\tilde{W}$ and so also has the SIBN property.

Since F is a direct limit of free right $Z[N]$-modules it is flat as a $Z[N]$-algebra. Therefore $F_G = F \otimes_{Z[N]} Q[G]$ is flat as a right $Z[G]$-module. We may define a multiplication which makes this module into a $Z[G]$-algebra by

$$(s_1^{-1}r_1 \otimes \alpha)(s_2^{-1}r_2 \otimes \beta) = s_1^{-1}r_1(\alpha s_2 \alpha^{-1})^{-1}(\alpha r_2 \alpha^{-1}) \otimes \alpha\beta$$

for r_1, r_2 in $Z[N]$, s_1, s_2 in $Z[N]\backslash\{0\}$ and α, β in G. We may also define an involution on F_G by $\overline{\gamma^{-1}} = (\bar{\gamma})^{-1}$.

The extended augmentation module $F_G \otimes_{Z[G]} Z = F_G \otimes_{Z[G]} Q$ is a Q-vector space, of dimension at most 1. However it is also a left D-vector space. As $Q[N]$ embeds in $F = M_n(D)$ and as N is infinite D has infinite dimension over Q. Therefore $F_G \otimes_{Z[G]} Z$ must be 0. That F_G has the SIBN property follows as in [Ro84] on using Theorem 4 of [Li91] instead of Rosset's 3.4. //

If N is torsion free we may argue that if $n \neq 1$ in N then $n - 1$ is a nonzero divisor in $Z[N]$ and so is invertible in F and hence in F_G. As it annihilates the augmentation module Z it follows that $F_G \otimes_{Z[G]} Z = 0$.

On the other hand, if $Z[\pi]$ has a safe extension Φ and the augmentation $Q[\pi]$-module Q has a finite free resolution F_* then on tensoring F_* with Φ we get an exact sequence of free Φ-modules, and hence $\chi(\pi) = 0$. In particular, the group ring of a noncyclic free group does not have a safe extension.

5. Ends and cohomology with free coefficients

A finitely generated group G has 0, 1, 2 or infinitely many ends. It has 0 ends if and only if it is finite, in which case $H^0(G;Z[G]) \cong Z$ and $H^q(G;Z[G]) = 0$ for $q > 0$. Otherwise $H^0(G;Z[G]) = 0$ and $H^1(G;Z[G])$ is a free abelian group of rank $e(G) - 1$, where $e(G)$ is the number of ends of G [Sp49]. The group G has more than one end if and only if it is either a nontrivial generalised free product with amalgamation $G \cong A *_C B$ or an HNN extension $A *_C \phi$ where C is a finite group. In particular, it has two ends if and only if it is virtually Z if and only if it has a (maximal) finite normal subgroup F such that the quotient G/F is either infinite cyclic (Z) or infinite dihedral ($D = (Z/2Z) * (Z/2Z)$). (See [DD]).

Lemma 8. *Let N be a finitely generated elementary amenable group with $h(N) > 1$. Then N has one end.*

Proof. Any group with infinitely many ends has nonabelian free subgroups, and so cannot be amenable. If $h(N) > 1$ then N is infinite and not virtually Z, and so must have one end. //

Let G be a group with a normal subgroup N, and let A be a left $Z[G]$-module. The *Lyndon-Hochschild-Serre spectral sequence* (LHSSS) for G as an extension of G/N by N and with coefficients the $Z[G]$-module A has E_2 term $H^p(G/N; H^q(N;A))$, r^{th} differential of bidegree $(r, 1-r)$ and converging to $H^{p+q}(G;A)$. (See Section 10.1 of [Mc]).

The argument of the next result is from [Ro75].

Theorem 9. *If G has a normal subgroup N which is the union of an increasing sequence of FP_r subgroups N_n such that $H^s(N_n;Z[N_n]) = 0$ for $s \leq r$ then $H^s(G;Z[G]) = 0$ for $s \leq r$.*

Proof. Let $s \leq r$. Since N_n is FP_r we have $H^s(N_n;Z[G]) = H^s(N_n;Z[N_n]) \otimes Z[G/N_n] = 0$. Let f be an s-cocycle for N with coefficients $Z[G]$, and let f_n denote the restriction of f to a cocycle on N_n. Then there is an $(s-1)$-cochain g_n on N_n such that $\delta g_n = f_n$. Since $\delta(g_{n+1}|_{N_n} - g_n) = 0$ and $H^{s-1}(N_n;Z[G]) = 0$ there is an $(s-2)$-cochain h_n on N_n with $\delta h_n = g_{n+1}|_{N_n} - g_n$. Choose an extension h'_n of h_n to N_{n+1} and let $\bar{g}_{n+1} = g_{n+1} - \delta h'_n$.

Then $\bar{g}_{n+1}|_{N_n} = g_n$ and $\delta\bar{g}_{n+1} = f_{n+1}$. In this way we may extend g_0 to an $(s-1)$-cochain g on N such that $f = \delta g$ and so $H^s(N; Z[G]) = 0$. The LHSSS for G as an extension of G/N by N, with coefficients $Z[G]$, now gives $H^s(G; Z[G]) = 0$ for $s \leq r$. //

Corollary. *If G has a normal subgroup N which is the union of an increasing sequence of finitely generated, one-ended subgroups then G has one end.* //

In particular, this corollary applies if N is a countable elementary amenable group and $h(N) > 1$, by Lemma 8. We can prove a vanishing theorem for higher cohomology groups under more restrictive assumptions on the normal subgroup.

Theorem 10. *If G has a countable locally nilpotent normal subgroup N such that $h(N) > r$ then $H^s(G; Z[G]) = 0$ for $s \leq r$.*

Proof. The subgroup N is the union of an increasing sequence of finitely generated nilpotent subgroups of Hirsch length $> r$. As finitely generated nilpotent groups are virtually poly-Z, Theorem 9 applies. //

Does this theorem remain true if N is assumed only to be an elementary amenable group with $h(N) > r$?

The second cohomology of a group with free coefficients ($H^2(G; Z[G])$) shall play an important role in our investigations. In particular, we would like to know when this group is 0, and when it is infinite cyclic. Unfortunately less is known about this group than about $H^1(G; Z[G])$. In [Fa74] Farrell has shown that if G is finitely presentable and has an element of infinite order then $H^2(G; Z[G])$ is either 0 or Z or is not finitely generated. He has also shown that if G is finitely presentable, has one end and $H^2(G; Z/2Z[G]) = Z/2Z$ then every finitely generated subgroup with one end has finite index in G (Proposition 2.4 of [Fa74]). (In particular, if G is torsion free then subgroups of infinite index in G are locally free). The heart of the argument involves material on the cohomology of a space relative to a family of supports. It would be of interest to have a purely algebraic exposition of this work of Farrell.