Chapter 1

The field of values

1.0 Introduction

Like the spectrum (or set of eigenvalues) $\sigma(\cdot)$, the field of values $F(\cdot)$ is a set of complex numbers naturally associated with a given n-by-n matrix A:

$$F(A) \equiv \{x^*Ax\colon\ x \in \mathbb{C}^n,\ x^*x = 1\}$$

The spectrum of a matrix is a discrete point set; while the field of values can be a continuum, it is always a compact convex set. Like the spectrum, the field of values is a set that can be used to learn something about the matrix, and it can often give information that the spectrum alone cannot give. The eigenvalues of Hermitian and normal matrices have especially pleasant properties, and the field of values captures certain aspects of this nice structure for general matrices.

1.0.1 Subadditivity and eigenvalues of sums

If only the eigenvalues $\sigma(A)$ and $\sigma(B)$ are known about two n-by-n matrices A and B, remarkably little can be said about $\sigma(A+B)$, the eigenvalues of the sum. Of course, $\text{tr}(A+B) = \text{tr}\,A + \text{tr}\,B$, so the sum of all the eigenvalues of $A+B$ is the sum of all the eigenvalues of A plus the sum of all the eigenvalues of B. But beyond this, nothing can be said about the eigenvalues of $A+B$ without more information about A and B. For example, even if all the eigenvalues of two n-by-n matrices A and B are known and fixed, the spectral radius of $A+B$ [the largest absolute value of an eigenvalue of $A+B$, denoted by $\rho(A+B)$] can be arbitrarily large (see Problem 1). On the other hand, if A and B are normal, then much can be said about the eigenvalues of $A + B$; for example, $\rho(A + B) \leq \rho(A) + \rho(B)$ in this case. Sums of matrices do arise in practice, and two relevant properties of the field of values $F(\cdot)$ are:

(a) The field of values is subadditive: $F(A + B) \subset F(A) + F(B)$, where the set sum has the natural definition of sums of all possible pairs, one from each; and
(b) The eigenvalues of a matrix lie inside its field of values: $\sigma(A) \subset F(A)$.

Combining these two properties yields the inclusions

$$\sigma(A + B) \subset F(A + B) \subset F(A) + F(B)$$

so if the two fields of values $F(A)$ and $F(B)$ are known, something can be said about the spectrum of the sum.

1.0.2 An application from the numerical solution of partial differential equations

Suppose that $A = [a_{ij}] \in M_n(\mathbb{R})$ satisfies

(a) A is tridiagonal ($a_{ij} = 0$ for $|i - j| > 1$), and

(b) $a_{i,i+1}a_{i+1,i} < 0$ for $i = 1, \ldots, n-1$.

Matrices of this type arise in the numerical solution of partial differential equations and in the analysis of dynamical systems arising in mathematical biology. In both cases, knowledge about the real parts of the eigenvalues of A is important. It turns out that rather good information about the eigenvalues of such a matrix can be obtained easily using the field of values $F(\cdot)$.

1.0.2.1 **Fact**: For any eigenvalue λ of a matrix A of the type indicated, we have

$$\min_{1 \leq i \leq n} a_{ii} \leq \operatorname{Re} \lambda \leq \max_{1 \leq i \leq n} a_{ii}$$

A proof of this fact is fairly simple using some properties of the field of values to be developed in Section (1.2). First, choose a diagonal matrix D

with positive diagonal entries such that $D^{-1}AD \equiv \hat{A} = [\hat{a}_{ij}]$ satisfies $\hat{a}_{ji} = -\hat{a}_{ij}$ for $j \neq i$. The matrix $D \equiv \text{diag}(d_1,\ldots, d_n)$ defined by

$$d_1 = 1, \text{ and } d_i = \left|\frac{a_{i,i-1}}{a_{i-1,i}}\right|^{\frac{1}{2}} d_{i-1},\ d_i > 0,\ i = 2,\ldots, n$$

will do. Since $\hat{A}$ and A are similar, their eigenvalues are the same. We then have

$$\begin{aligned}\text{Re } \sigma(A) = \text{Re } \sigma(\hat{A}) \subset \text{Re } F(\hat{A}) &= F(\tfrac{1}{2}(\hat{A} + \hat{A}^T))\\ &= F(\text{diag}(a_{11},\ldots, a_{nn}))\\ &= \text{Convex hull of } \{a_{11},\ldots, a_{nn}\} = [\min_i a_{ii}, \max_i a_{ii}]\end{aligned}$$

The first inclusion follows from the spectral containment property (1.2.6), the next equality follows from the projection property (1.2.5), the next equality follows from the special form achieved for $\hat{A}$, and the last equality follows from the normality property (1.2.9) and the fact that the eigenvalues of a diagonal matrix are its diagonal entries. Since the real part of each eigenvalue $\lambda \in \sigma(A)$ is a convex combination of the main diagonal entries a_{ii}, $i = 1,\ldots, n$, the asserted inequalities are clear and the proof is complete.

1.0.3 Stability analysis

In an analysis of the stability of an equilibrium in a dynamical system governed by a system of differential equations, it is important to know if the real part of every eigenvalue of a certain matrix A is negative. Such a matrix is called *stable*. In order to avoid juggling negative signs, we often work with *positive stable* matrices (all eigenvalues have positive real parts). Obviously, A is positive stable if and only if $-A$ is stable. An important sufficient condition for a matrix to be positive stable is the following fact.

1.0.3.1 Fact: Let $A \in M_n$. If $A + A^*$ is positive definite, then A is positive stable.

This is another application of properties of the field of values $F(\cdot)$ to be developed in Section (1.2). By the spectral containment property (1.2.6), $\text{Re } \sigma(A) \subset \text{Re } F(A)$, and, by the projection property (1.2.5), $\text{Re } F(A) =$

$F(\frac{1}{2}(A + A^*))$. But, since $A + A^*$ is positive definite, so is $\frac{1}{2}(A + A^*)$, and hence, by the normality property (1.2.9), $F(\frac{1}{2}(A + A^*))$ is contained in the positive real axis. Thus, each eigenvalue of A has a positive real part, and A is positive stable.

Actually, more is true. If $A + A^*$ is positive definite, and if $P \in M_n$ is any positive definite matrix, then PA is positive stable because

$$(P^{\frac{1}{2}})^{-1}[PA]P^{\frac{1}{2}} = P^{\frac{1}{2}}AP^{\frac{1}{2}}, \text{ and}$$

$$P^{\frac{1}{2}}AP^{\frac{1}{2}} + (P^{\frac{1}{2}}AP^{\frac{1}{2}})^* = P^{\frac{1}{2}}(A + A^*)P^{\frac{1}{2}}$$

where $P^{\frac{1}{2}}$ is the unique (Hermitian) positive definite square root of P. Since congruence preserves positive definiteness, the eigenvalues of PA have positive real parts for the same reason as A. Lyapunov's theorem (2.2.1) shows that all positive stable matrices arise in this way.

1.0.4 An approximation problem

Suppose we wish to approximate a given matrix $A \in M_n$ by a complex multiple of a Hermitian matrix of rank at most one, as closely as possible in the Frobenius norm $\|\cdot\|_2$. This is the problem

$$\text{minimize } \|A - cxx^*\|_2^2 \text{ for } x \in \mathbb{C}^n \text{ with } x^*x = 1 \text{ and } c \in \mathbb{C} \qquad (1.0.4.1)$$

Since the inner product $[A,B] \equiv \text{tr } AB^*$ generates the Frobenius norm, we have

$$\|A - cxx^*\|_2^2 = [A - cxx^*, A - cxx^*]$$

$$= \|A\|_2^2 - 2 \text{ Re } \bar{c}\,[A,xx^*] + |c|^2$$

which, for a given unit vector x, is minimized by $c = [A,xx^*]$. Substitution of this value into (1.0.4.1) transforms our problem into

$$\text{minimize } (\|A\|_2^2 - |[A,xx^*]|^2) \text{ for } x \in \mathbb{C}^n \text{ with } x^*x = 1$$

or, equivalently,

$$\text{maximize } |[A,xx^*]| \text{ for } x \in \mathbb{C}^n \text{ with } x^*x = 1$$

A vector x_0 that solves the latter problem (we are maximizing a continuous function on a compact set) will yield a rank one solution $[A, x_0x_0^*]x_0x_0^*$ to our original problem. Since $[A, xx^*] = \text{tr } Axx^* = x^*Ax$, we are led naturally to finding a unit vector x such that the point x^*Ax in the field of values $F(A)$ has maximum distance from the origin. The absolute value of such a point is called the *numerical radius* of A [often denoted by $r(A)$] by analogy with the *spectral radius* [often denoted by $\rho(A)$], which is the absolute value of a point in the spectrum $\sigma(A)$ that is at maximum distance from the origin.

Problems

1. Consider the real matrices

$$A = \begin{bmatrix} 1-\alpha & 1 \\ \alpha(1-\alpha)-1 & \alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} 1+\alpha & 1 \\ -\alpha(1+\alpha)-1 & -\alpha \end{bmatrix}$$

Show that $\sigma(A)$ and $\sigma(B)$ are independent of the value of $\alpha \in \mathbb{R}$. What are they? What is $\sigma(A+B)$? Show that $\rho(A+B)$ is unbounded as $\alpha \to \infty$.

2. In contrast to Problem 1, show that if $A, B \in M_n$ are normal, then $\rho(A+B) \leq \rho(A) + \rho(B)$.

3. Show that "$<$" in (1.0.2(b)) may be replaced by "$\leq$," the main diagonal entries a_{ii} may be complex, and Fact (1.0.2.1) still holds if a_{ii} is replaced by Re a_{ii}.

4. Show that the problem of approximating a given $A \in M_n$ by a positive semidefinite rank one matrix with spectral radius one can be solved if one can find a unit vector x such that the point x^*Ax in $F(A)$ is furthest to the right in the complex plane, that is, Re x^*Ax is maximized.

1.1 Definitions

In this section we define the field of values and certain related objects.

1.1.1 Definition. The *field of values* of $A \in M_n$ is

$$F(A) \equiv \{x^*Ax: x \in \mathbb{C}^n, x^*x = 1\}$$

Thus, $F(\cdot)$ is a function from M_n into subsets of the complex plane.

$F(A)$ is just the normalized locus of the Hermitian form associated with A. The field of values is often called the *numerical range*, especially in the context of its analog for operators on infinite dimensional spaces.

Exercise. Show that $F(I) = \{1\}$ and $F(\alpha I) = \{\alpha\}$ for all $\alpha \in \mathbb{C}$. Show that $F\begin{bmatrix}1&0\\0&0\end{bmatrix}$ is the closed unit interval $[0,1]$, and $F\begin{bmatrix}0&2\\0&0\end{bmatrix}$ is the closed unit disc $\{z \in \mathbb{C}: |z| \leq 1\}$.

The field of values $F(A)$ may also be thought of as the image of the surface of the Euclidean unit ball in $\mathbb{C}^n$ (a compact set) under the continuous transformation $x \to x^*Ax$. As such, $F(A)$ is a compact (and hence bounded) set in $\mathbb{C}$. An unbounded analog of $F(\cdot)$ is also of interest.

1.1.2 Definition. The *angular field of values* is

$$F'(A) \equiv \{x^*Ax: x \in \mathbb{C}^n, x \neq 0\}$$

Exercise. Show that $F'(A)$ is determined geometrically by $F(A)$; every open ray from the origin that intersects $F(A)$ in a point other than the origin is in $F'(A)$, and $0 \in F'(A)$ if and only if $0 \in F(A)$. Draw a typical picture of an $F(A)$ and $F'(A)$ assuming that $0 \notin F(A)$.

It will become clear that $F'(A)$ is an angular sector of the complex plane that is anchored at the origin (possibly the entire complex plane). The angular opening of this sector is of interest.

1.1.3 Definition. The *field angle* $\Theta \equiv \Theta(A) \equiv \Theta(F'(A)) \equiv \Theta(F(A))$ of $A \in M_n$ is defined as follows:

(a) If 0 is an interior point of $F(A)$, then $\Theta(A) \equiv 2\pi$.

(b) If 0 is on the boundary of $F(A)$ and there is a (unique) tangent to the boundary of $F(A)$ at 0, then $\Theta(A) \equiv \pi$.

(c) If $F(A)$ is contained in a line through the origin, $\Theta(A) \equiv 0$.

(d) Otherwise, consider the two different support lines of $F(A)$ that go through the origin, and let $\Theta(A)$ be the angle subtended by these two lines at the origin. If $0 \notin F(A)$, these support lines will be uniquely determined; if 0 is on the boundary of $F(A)$, choose the two support lines that give the minimum angle.

We shall see that $F(A)$ is a compact convex set for every $A \in M_n$, so this informal definition of the field angle makes sense. The field angle is just the angular opening of the smallest angular sector that includes $F(A)$, that is, the angular opening of the sector $F'(A)$.

Finally, the size of the bounded set $F(A)$ is of interest. We measure its size in terms of the radius of the smallest circle centered at the origin that contains $F(A)$.

1.1.4 Definition. The *numerical radius* of $A \in M_n$ is

$$r(A) \equiv \max\{|z|: z \in F(A)\}$$

The numerical radius is a vector norm on matrices that is not a matrix norm (see Section (5.7) of [HJ]).

Problems

1. Show that among the vectors entering into the definition of $F(A)$, only vectors with real nonnegative first coordinate need be considered.

2. Show that both $F(A)$ and $F'(A)$ are simply connected for any $A \in M_n$.

3. Show that for each $0 \leq \theta \leq \pi$, there is an $A \in M_2$ with $\Theta(A) = \theta$. Is $\Theta(A) = 3\pi/2$ possible?

4. Why is the "max" in (1.1.4) attained?

5. Show that the following alternative definition of $F(A)$ is equivalent to the one given:

$$F(A) \equiv \{x^*Ax/x^*x: \ x \in \mathbb{C}^n \text{ and } x \neq 0\}$$

Thus, $F(\cdot)$ is a normalized version of $F'(\cdot)$.

6. Determine $F\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $F\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $F\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

7. If $A \in M_n$ and $\alpha \in F(A)$, show that there is a unitary matrix $U \in M_n$ such that α is the 1,1 entry of U^*AU.

8. Determine as many different possible types of sets as you can that can be an $F'(A)$.

9. Show that $F(A^*) = \overline{F(A)}$ and $F'(A^*) = \overline{F'(A)}$ for all $A \in M_n$.

10. Show that all of the main diagonal entries and eigenvalues of a given $A \in M_n$ are in its field of values $F(A)$.

1.2 Basic properties of the field of values

As a function from M_n into subsets of $\mathbb{C}$, the field of values $F(\cdot)$ has many useful functional properties, most of which are easily established. We catalog many of these properties here for reference and later use. The important property of convexity is left for discussion in the next section.

The sum or product of two subsets of $\mathbb{C}$, or of a subset of $\mathbb{C}$ and a scalar, has the usual algebraic meaning. For example, if $S, T \subset \mathbb{C}$, then $S + T \equiv \{s + t: s \in S, t \in T\}$.

1.2.1 Property: *Compactness*. For all $A \in M_n$,

$$F(A) \text{ is a compact subset of } \mathbb{C}$$

Proof: The set $F(A)$ is the range of the continuous function $x \rightarrow x^*Ax$ over the domain $\{x: x \in \mathbb{C}^n, x^*x = 1\}$, the surface of the Euclidean unit ball, which is a compact set. Since the continuous image of a compact set is compact, it follows that $F(A)$ is compact. □

1.2.2 Property: *Convexity*. For all $A \in M_n$,

$$F(A) \text{ is a convex subset of } \mathbb{C}$$

The next section of this chapter is reserved for a proof of this fundamental fact, known as the *Toeplitz-Hausdorff theorem*. At this point, it is clear that $F(A)$ must be a *connected* set since it is the continuous image of a connected set.

Exercise. If A is a diagonal matrix, show that $F(A)$ is the convex hull of the diagonal entries (the eigenvalues) of A.

The field of values of a matrix is changed in a simple way by adding a scalar multiple of the identity to it or by multiplying it by a scalar.

1.2.3 **Property**: *Translation*. For all $A \in M_n$ and $\alpha \in \mathbb{C}$,

$$F(A + \alpha I) = F(A) + \alpha$$

Proof: We have $F(A+\alpha I) = \{x^*(A+\alpha I)x: x^*x = 1\} = \{x^*Ax + \alpha x^*x: x^*x = 1\} = \{x^*Ax + \alpha: x^*x = 1\} = \{x^*Ax: x^*x = 1\} + \alpha = F(A) + \alpha$. □

1.2.4 **Property**: *Scalar multiplication*. For all $A \in M_n$ and $\alpha \in \mathbb{C}$,

$$F(\alpha A) = \alpha F(A)$$

Exercise. Prove property (1.2.4) by the same method used in the proof of (1.2.3).

For $A \in M_n$, $\mathrm{H}(A) \equiv \frac{1}{2}(A + A^*)$ denotes the *Hermitian part of* A and $\mathrm{S}(A) \equiv \frac{1}{2}(A - A^*)$ denotes the *skew-Hermitian part of* A; notice that $A = \mathrm{H}(A) + \mathrm{S}(A)$ and that $\mathrm{H}(A)$ and $i\mathrm{S}(A)$ are both Hermitian. Just as taking the real part of a complex number projects it onto the real axis, taking the Hermitian part of a matrix projects its field of values onto the real axis. This simple fact helps in locating the field of values, since, as we shall see, it is relatively easy to deal with the field of values of a Hermitian matrix. For a set $S \subset \mathbb{C}$, we interpret Re S as $\{\mathrm{Re}\, s: s \in S\}$, the projection of S onto the real axis.

1.2.5 **Property**: *Projection*. For all $A \in M_n$,

$$F(\mathrm{H}(A)) = \mathrm{Re}\, F(A)$$

Proof: We calculate $x^*\mathrm{H}(A)x = x^*\frac{1}{2}(A + A^*)x = \frac{1}{2}(x^*Ax + x^*A^*x) = \frac{1}{2}(x^*Ax + (x^*Ax)^*) = \frac{1}{2}(x^*Ax + \overline{x^*Ax}) = \mathrm{Re}\, x^*Ax$. Thus, each point in $F(\mathrm{H}(A))$ is of the form Re z for some $z \in F(A)$ and vice versa. □

We denote the open upper half-plane of $\mathbb{C}$ by $UHP \equiv \{z \in \mathbb{C}: \mathrm{Im}\, z > 0\}$, the open left half-plane of $\mathbb{C}$ by $LHP \equiv \{z \in \mathbb{C}: \mathrm{Re}\, z < 0\}$, the open right half-plane of $\mathbb{C}$ by $RHP \equiv \{z \in \mathbb{C}: \mathrm{Re}\, z > 0\}$, and the closed right half-plane of $\mathbb{C}$ by $RHP_0 \equiv \{z \in \mathbb{C}: \mathrm{Re}\, z \geq 0\}$. The projection property gives a simple indication of when $F(A) \subset RHP$ or RHP_0 in terms of positive definiteness or positive semidefiniteness.

1.2.5a **Property**: *Positive definite indicator function.* Let $A \in M_n$. Then $F(A) \subset RHP$ if and only if $A + A^*$ is positive definite

1.2.5b **Property**: *Positive semidefinite indicator function.* Let $A \in M_n$. Then

$$F(A) \subset RHP_0 \text{ if and only if } A + A^* \text{ is positive semidefinite}$$

Exercise. Prove (1.2.5a) and (1.2.5b) (the proofs are essentially the same) using (1.2.5) and the definition of positive definite and semidefinite (see Chapter 7 of [HJ]).

The point set of eigenvalues of $A \in M_n$ is denoted by $\sigma(A)$, the *spectrum* of A. A very important property of the field of values is that it includes the eigenvalues of A.

1.2.6 **Property**: *Spectral containment.* For all $A \in M_n$,

$$\sigma(A) \subset F(A)$$

Proof: Suppose that $\lambda \in \sigma(A)$. Then there exists some nonzero $x \in \mathbb{C}^n$, which we may take to be a unit vector, for which $Ax = \lambda x$ and hence $\lambda = \lambda x^* x = x^*(\lambda x) = x^* A x \in F(A)$. □

Exercise. Use the spectral containment property (1.2.6) to show that the eigenvalues of a positive definite matrix are positive real numbers.

Exercise. Use the spectral containment property (1.2.6) to show that the eigenvalues of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are imaginary.

The following property underlies the fact that the numerical radius is a vector norm on matrices and is an important reason why the field of values is so useful.

1.2.7 **Property**: *Subadditivity.* For all $A, B \in M_n$,

$$F(A + B) \subset F(A) + F(B)$$

Proof: $F(A + B) = \{x^*(A + B)x\colon x \in \mathbb{C}^n,\ x^*x = 1\} = \{x^*Ax + x^*Bx\colon x \in \mathbb{C}^n,$