On Complex Projective Hypersurfaces which are Homology-Pn's

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Dedicated to Friedrich Hirzebruch

Abstract. We discuss hypersurfaces in $\mathbf{P}_{n+1}$ that are homology-$\mathbf{P}_n$'s, i.e., they have the integral homology of $\mathbf{P}_n$. The only cohomology ring-$\mathbf{P}_n$'s are the hyperplanes. Using singularity theoretic methods, we construct examples of homology-$\mathbf{P}_n$'s with isolated singularities in any dimension $n \geq 2$ and for any degree $d \geq 3$. In the odd-dimensional case, these are topological manifolds. Our methods yield interesting examples in singularity theory, e.g., isolated hypersurface singularity links that are topological spheres, but that are not associated to polynomials of the familiar "Pham-Brieskorn" type. Furthermore, we classify normal homology-$\mathbf{P}_2$'s in $\mathbf{P}_3$ with $\mathbf{C}^*$-action up to isomorphy and homology-$\mathbf{P}_3$'s in $\mathbf{P}_4$ with isolated singularities up to homeomorphy, and we construct examples of homology-$\mathbf{P}_n$'s with non-isolated singularities.

1: INTRODUCTION AND STATEMENT OF RESULTS

Considering the importance of the complex projective n-space $\mathbf{P}_n = \mathbf{P}_n(\mathbf{C})$ in algebraic geometry and topology, it is obvious that characterizing that space by algebro-geometric or topological properties always has been a matter of great interest. Therefore, it is quite natural to investigate spaces that share some of these properties. In this paper, we look for hypersurfaces in $\mathbf{P}_{n+1}$ (where $n \geq 2$) with normal or even isolated singularities which are *homology-$\mathbf{P}_n$'s*, i.e., which have the same integral homology as $\mathbf{P}_n$. As the integral homology $H_*(V, \mathbf{Z})$ of any n-dimensional projective variety V always contains a graded subgroup isomorphic to $H_*(\mathbf{P}_n, \mathbf{Z})$, homology-$\mathbf{P}_n$'s are characterized among such varieties by an obvious minimality property. By the universal coefficient formula, the property of being an homology-$\mathbf{P}_n$ is equivalent to having the integral cohomology *groups* of $\mathbf{P}_n$. The condition that the integral cohomology *ring* agrees with that of $\mathbf{P}_n$ turns out to be rather restrictive, as shows our first result:

Theorem 1. *(Cohomology Ring-$\mathbf{P}_n$'s are Actual $\mathbf{P}_n$'s.) Let V be a closed subvariety of dimension $\dim V = n \geq 2$ in some projective space $\mathbf{P}_N$ which can be described by a system of at most $N-2$ homogeneous polynomials. If the cohomology rings $H^\bullet(V, \mathbf{Z})$ and $H^\bullet(\mathbf{P}_n, \mathbf{Z})$ are isomorphic as abstract graded rings, then V is a linear subspace of $\mathbf{P}_N$.*

Actually, the condition on the cohomology ring structure can be slightly weakened; see the precise statement in section 1 below. Note that the condition on the number of defining equations is always satisfied for complete intersection varieties, and only for these in the surface case $n = 2$. The example of the Veronese surface V (i.e., the projective plane embedded by quadratic forms in $\mathbf{P}_5$) shows that this condition is sharp, as V just can be described by 4 quadratic polynomials. Moreover, the example of the smooth plane quadric curve (i.e., the projective line embedded by quadratic forms in $\mathbf{P}_2$) shows that the condition $\dim V = n \geq 2$ is sharp, too. (Of course, in the case of curves, the number of defining equations is at least $N - 1$).

Turning now to the study of homology-$\mathbf{P}_n$'s, our main results are as follows:

Theorem 2. *(Examples of Homology-$\mathbf{P}_n$'s with Isolated Singularities.) For any dimension $n \geq 2$, degree $d \geq 3$, and integer a with $1 \leq a < d-1$, we consider the hypersurface $V := V^a_{n,d} : (f_{d,a} = 0)$ in $\mathbf{P}_{n+1}$ defined by*

$$f_{d,a}(x_0, x_1, \ldots, x_n, x_{n+1}) := x_0^a x_1^{d-a} + x_1 x_2^{d-1} + \ldots + x_{n-1} x_n^{d-1} + x_{n+1}^d \; .$$

This hypersurface has isolated singularities and satisfies
(i) $H_\bullet(V, \mathbf{Q}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Q})$ for $(a, d) = 1$;
(ii) $H_\bullet(V, \mathbf{Z}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Z})$ for $(a, d) = (a, d-1) = 1$.

The proof uses singularity theoretic arguments; it is given in section 2. It provides examples for the following phenomena that may be of interest in singularity theory and topology:

(i) Examples of hypersurface singularities with one-dimensional singular locus and with monodromy operator equal to the identity (section 2, (ii), Lemma 2 and Remark). This contrasts with the situation for isolated hypersurface singularities, as described by A'Campo.

(ii) Examples of isolated complex hypersurface singularity links in all real dimensions $2n - 1 \geq 5$ which are integral homology spheres (and hence topological spheres), defined by positively weighted homogeneous polynomials which are not equivalent to polynomials of the familiar "Pham-Brieskorn" type $\sum_{j=0}^n x_j^{a_j}$ (see section 2, Corollary 1). This contrasts with the situation in dimension 2 (see section 3, Appendix).

(iii) Examples of projective hypersurfaces in all odd dimensions $n \geq 3$ with (one or two) isolated singularities and with minimal homology which are topological manifolds. (In fact, if an odd-dimensional projective hypersurface with isolated singularities has the integral homology of $\mathbf{P}_n$, then it is a topological manifold; see section 2, (v), Proposition. Such a variety even has the integral cohomology groups and the rational homotopy type of $\mathbf{P}_n$, but is not homotopy equivalent to $\mathbf{P}_n$, e.g., by Theorem 1.) Again, this contrasts with the situation in dimension 2: By a famous result of Mumford, a surface with normal singularities (e.g., a complete intersection surface with isolated singularities) never is a topological manifold.

The cases $n = 1$ or $d = 2$ not covered by the theorem are easy to deal with: The plane curve $V^a_{1,d}$ defined by $x_0^a x_1^{d-a} + x_2^d$ for $0 < a < d$ and $(a, d) = 1$ actually is homeomorphic to the projective line. For degree $d = 2$, it is well known that the only homology-$\mathbf{P}_n$'s among the quadric hypersurfaces with isolated singularities are the even-dimensional quadratic cones, and that, moreover, the odd-dimensional smooth quadrics Q_n are the only non-linear smooth hypersurfaces which are homology-$\mathbf{P}_n$'s.

After these examples, we turn to classification results for surfaces and threefolds. Note that the hypersurfaces $V^a_{n,d}$ admit a natural algebraic $\mathbf{C}^*$-action, as the affine equation at $\mathfrak{o}_0 := (1:0:0:\ldots:0)$ (and also at $\mathfrak{o}_1 := (0:1:0:\ldots:0)$) is weighted homogeneous. It turns out that in the class of normal surfaces in $\mathbf{P}_3$ with such a $\mathbf{C}^*$-action, our $V^a_{2,d}$ are the only homology-$\mathbf{P}_2$'s; moreover, they are pairwise non homeomorphic. In the case of threefolds, the topological type of an arbitrary homology-$\mathbf{P}_3$ in $\mathbf{P}_4$ with isolated singularities is completely determined by the degree, so such hypersurfaces with analytically different singularities may be homeomorphic. The precise statement is given in the following result (see section 3):

Theorem 3. *(Classification Results for Homology-$\mathbf{P}_2$'s and -$\mathbf{P}_3$'s.)*
($n = 2$) Let V be a homology-$\mathbf{P}_2$ in $\mathbf{P}_3$ of degree $d \geq 3$ with isolated singularities which admits an algebraic $\mathbf{C}^*$-action. Then V is equal to $V^a_{2,d}$ for a unique integer a satisfying $1 \leq a < d-1$ and $(a, d-1) = (a, d) = 1$. These surfaces are pairwise non homeomorphic.
($n = 3$) Let V, V' be homology-$\mathbf{P}_3$'s in $\mathbf{P}_4$ with isolated singularities. Then V and V' are homeomorphic if and only if they have the same degree.

Examples of homology-$\mathbf{P}_n$'s in dimensions $n \geq 3$ with singular locus of positive dimension can be obtained by more elementary methods than in the isolated singularity case. Such examples will be presented in section 4 (see Theorem 4).

We mention that Theorem 1 in the hypersurface case and some of the examples in Theorem 2 in the two-dimensional case (namely, the case $a = 1$, slightly disguised) have already appeared in [ChDi]. Moreover, the varieties $V^1_{n,d}$ play a key rôle in the work of Libgober on the connected sum decomposition of smooth hypersurfaces, and the fact that they have the integral homology of $\mathbf{P}_n$ is stated in [LiWo: § 2]. — Smooth projective manifolds with the rational cohomology ring of $\mathbf{P}_n$ have been investigated by several authors. For results, mainly for $n = 4$, and further references, the reader is referred to Wilson's article.

It is a pleasure for us to thank Ludger Kaup for his stimulating interest. In particular, section 1 was strongly influenced by him through discussions with one of us. Moreover, in sections 3 and 4, we closely follow ideas of earlier joint work of his and the first-named author. His remarks on a preliminary version of the present paper were most helpful. We would also like to thank Karl Fieseler who carefully read the text. Finally, we thank the referee for his valuable comments and his indications of some rather fine inexactitudes. His pertinent remarks encouraged us to keep the paper accessible to non-specialists. For them, the recent book [STH] might be useful as an introduction to and reference for many results and techniques used in the sequel.

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NOTATIONS AND CONVENTIONS

Most of the varieties to be considered in the sequel are—in suitable affine coordinates—defined by *weighted homogeneous* (or quasihomogeneous) polynomials. Recall that by definition, such a polynomial $p(y_0, y_1, \dots, y_m)$ satisfies an identity $p(t^{q_0}y_0, t^{q_1}y_1, \dots, t^{q_m}y_m) = t^N \cdot p(y_0, y_1, \dots, y_m)$ for a suitable vector $\mathfrak{q} = (q_0, q_1, \dots, q_m)$ of integers q_j and an integer $N =: \mathfrak{q}\text{-deg}(p)$, the $\mathfrak{q}$-degree, so p is a homogeneous element of degree $\mathfrak{q}\text{-deg}(p) = N$ with respect to the grading of the polynomial algebra $\mathbf{C}[y_0, y_1, \dots, y_m]$ given by $\mathfrak{q}\text{-deg}(y_j) = q_j$. We do not assume that the q_j's are necessarily positive; in fact, the "mixed" case occurs in the proof of Lemma 2 in section 2.

We adopt here the convention to call the q_j's the *weights*. They are just the weights of the $\mathbf{C}^*$-action on $\mathbf{C}^{m+1}$ given by

$$t_\bullet(y_0, y_1, \dots, y_m) = (t^{q_0}y_0, t^{q_1}y_1, \dots, t^{q_m}y_m)$$

that is associated to the grading. Dividing by the greatest common divisor of the weights if necessary, we may always assume that the weight vector $\mathfrak{q}$ is primitive, i.e., $\gcd(q_0, q_1, \dots, q_m) = 1$ or, equivalently, that the action is effective. We sometimes call p a $\mathfrak{q}$-homogeneous polynomial. The pair $(\mathfrak{q}, \mathfrak{q}\text{-deg}(p))$ is called the *type* of p.

Concerning the notion of "weight", there are different conventions used in the literature, especially in the case of a strictly positive grading (i.e., all $q_j > 0$, corresponding to a "good" $\mathbf{C}^*$-action). In addition to conventions discussed in [TRCS: Ch. 7, § 1], we mention the one adopted in several papers (by Milnor, Orlik, and some others, including earlier papers of the first-named author) where the positive rational numbers $w_j := \mathfrak{q}\text{-deg}(f)/q_j$ are called weights. Instead, we call these w_j the *coweights* in the sequel. If we want to emphasize that we are in the case of a strictly positive grading, we sometimes call such a $\mathfrak{q}$-homogeneous polynomial *positively* weighted homogeneous.

References [..] to articles in journals, conference proceedings, etc., are given by the Names of author(s) (or abbreviations), using Small Capitals. References to monographs are given by Title Initials in Capitals [TIC]. The sign [-] refers to the last preceding reference.

1: PROJECTIVE VARIETIES WITH THE COHOMOLOGY RING OF $\mathbf{P}_N$.

In this section, we prove the result mentioned in the introduction above: A hypersurface with the same integral cohomology ring as $\mathbf{P}_n$ is a hyperplane, so it just *is* $\mathbf{P}_n$. The actual, slightly more general, statement is as follows.

Theorem 1. *Let V be a closed subvariety of dimension $\dim V = n \geq 2$ in some projective space $\mathbf{P}_N$ which (set-theoretically) can be described by a system of at most $N-2$ homogeneous polynomials. If the cohomology group $H^2(V,\mathbf{Z})$ is generated (up to torsion) by a class u such that u^n generates $H^{2n}(V,\mathbf{Z})$, then V is a linear subspace of $\mathbf{P}_N$.*

Proof. Denote with $j : V \hookrightarrow \mathbf{P}_N$ the inclusion mapping and with $\omega \in H^2(\mathbf{P}_N)$ the canonical generator. Then there is an integer $\alpha \neq 0$ (without loss of generality $\alpha > 0$) such that $j^*\omega = \alpha u$ holds in $H^2(V)$ (up to torsion), and hence $j^*\omega^n = \alpha^n u^n$ holds in $H^{2n}(V)$. By a well known property of the degree (see, e.g., [PAG: p. 171]), we have

$$\deg V = \langle \omega^n, j_*[V] \rangle = \langle j^*\omega^n, [V] \rangle = \alpha^n \langle u^n, [V] \rangle = \alpha^n$$

(where $\langle -, - \rangle$ denotes the usual pairing between homology and cohomology, sometimes called "Kronecker product"). In order to show that $\alpha = 1$, look at the following part of the exact cohomology sequence of the pair $(\mathbf{P}_N, V)$:

$$H^2(\mathbf{P}_N) \xrightarrow{j^*} H^2(V) \xrightarrow{\delta^*} H^3(\mathbf{P}_N, V) .$$

By Lefschetz duality, the last group is isomorphic to $H_{2N-3}(\mathbf{P}_N \setminus V)$. As V can be defined by at most $N-2$ equations ($f_j = 0$), the complex manifold $\mathbf{P}_N \setminus V$ is the union of at most $N-2$ affine open subsets ($f_j \neq 0$) and thus is *topologically* $(N-3)$*-complete* (see Fieseler and Kaup [$-_1$: § 1 and 2.3] for the definition of topological completeness and for the result). It follows from the theorem stated in the introduction of [$-_2$] that $H_{2N-3}(\mathbf{P}_N \setminus V)$ has no torsion (note that we may assume $N \geq 3$, so $2N-3 \geq N$ holds). Using the exactness of the sequence, we infer that $H^2(V)$ is free, generated by u, and that δ^* is the zero homomorphism as $\delta^* j^*\omega = \alpha\delta^* u$ vanishes. Hence $j^*\omega$

is a generator of $H^2(V)$, which yields $\alpha = 1$ and thus $\deg V = 1$. •

Remark. The first part of the previous argument proves the well-known fact that cohomology ring of V contains the truncated polynomial ring

$$\mathbf{Z}[j^*\omega]/(j^*\omega^{n+1}) \cong H^*(\mathbf{P}_n, \mathbf{Z})$$

as a graded subring. — The notion of a topologically q-complete space used in the second part is modeled after the topological properties of *analytically* q-complete spaces. In fact, by a theorem of Hamm, an analytically q-complete complex space of dimension n has the homotopy type of a CW-complex of (topological) dimension at most $n+q$. The topological completeness has a much nicer behaviour and better permanence properties with respect to standard operations; in particular, it is a homeomorphy invariant.

2: PROJECTIVE HYPERSURFACES WITH ISOLATED SINGULARITIES WHICH ARE HOMOLOGY-$\mathbf{P}_N$'S.

In this section, we prove Theorem 2 as stated in the introduction. In order to show that those hypersurfaces $V^a_{n,d} : (f_{d,a} = 0)$ in $\mathbf{P}_{n+1}$ (with $n \geq 2$) which satisfy the conditions on d and a have the integral homology of $\mathbf{P}_n$, we proceed in several steps. First, in steps *(i)–(iii)*, we use duality and monodromy arguments to check that they are rational homology-$\mathbf{P}_n$'s. In *(iv)*, we state necessary and sufficient conditions (in terms of Milnor lattices) for a rational homology-$\mathbf{P}_n$ in $\mathbf{P}_{n+1}$ with isolated singularities to be an integral homology-$\mathbf{P}_n$. That turns out to have interesting consequences, and we briefly digress to discuss some of them in *(v)* (e.g., in odd dimensions, such a variety is a topological manifold; see the Proposition below). Finally, using results of Milnor, Orlik, and Randell on the monodromy of certain weighted homogeneous singularities, we show in *(vi)–(viii)* that our examples satisfy these conditions.

(i) We begin with a characterization of *rational* homology-$\mathbf{P}_n$'s in terms of the monodromy operator of the defining homogeneous polynomial f (for the moment, we do not assume $f = f_{d,a}$). Let $V : (f = 0)$ be a hypersurface of degree $d \geq 2$ in $\mathbf{P}_{n+1}$.

Lemma 1. *The following statements are equivalent:*
(α) $H_\bullet(V, \mathbf{Q}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Q})$;
(β) Let $F \subset \mathbf{C}^{n+2}$ *be the Milnor fibre* $(f = 1)$ *of* f, *and let* $h_f^* : \tilde{H}^\bullet(F, \mathbf{Q}) \to \tilde{H}^\bullet(F, \mathbf{Q})$ *be the monodromy operator. Then all eigenvalues of* h_f^* *are different from* 1.

Proof (cf. [Oka$_2$: Thm.1, proof]). Dualizing the argument used in the proof of Theorem 1, we infer from the exact homology sequence of the pair $(\mathbf{P}_{n+1}, V)$ and from the relative (Alexander-Lefschetz type) duality theorem that statement (α) is equivalent to the vanishing of $\tilde{H}^\bullet(\mathbf{P}_{n+1} \setminus V, \mathbf{Q})$. The affine variety $\mathbf{P}_{n+1} \setminus V$ is the quotient of $\mathbf{C}^{n+2} \setminus (f = 0)$ under the standard action of $\mathbf{C}^*$ by multiplication. By homogeneity, the surjective mapping $f| : \mathbf{C}^{n+2} \setminus (f = 0) \to \mathbf{C}^*$ is equivariant with respect to the action $(\tau, t) \mapsto \tau^d \cdot t$ on the base. Hence, the stabilizer subgroup of the Milnor fibre F (and of any other fibre) is the group C_d of d-th roots of unity, so the orbit space F/C_d is canonically identified with $\mathbf{P}_{n+1} \setminus V$. The total space $E : (|f| = 1)$ of the Milnor fibration $f|_E : E \to S^1$ is invariant under the free S^1-action that is induced from the standard action of $\mathbf{C}^*$, and the action of the standard generator $\zeta := \exp(2\pi i/d)$ of C_d on $F \subset \mathbf{C}^{n+2}$ given by $(x_0, \ldots, x_{n+1}) \mapsto (\zeta x_0, \ldots, \zeta x_{n+1})$ is the natural geometric monodromy h_f of the Milnor fibration. Hence, the reduced cohomology of $\mathbf{P}_{n+1} \setminus V$ is isomorphic to $\tilde{H}^\bullet(F, \mathbf{Q})^{h_f^*}$, the fixed part under h_f^*, and the latter is $\ker(\mathrm{id} - h_f^*)$, the eigenspace of 1. •

(ii) To obtain examples of homogeneous polynomials $f(x_0, x_1, \ldots, x_n, x_{n+1})$ that satisfy condition (β) from above, we first search for a homogeneous polynomial $g(x_0, x_1, \ldots, x_n)$ with trivial monodromy. Adding to g the monomial x_{n+1}^d with $d := \deg g$ will yield f which has the required properties by a Thom-Sebastiani type argument due to Oka (see the proof of Lemma 3 below). So let us consider the homogeneous polynomial

$$g = g_{d,a}(x_0, x_1, \ldots, x_n) := x_0^a x_1^{d-a} + x_1 x_2^{d-1} + \ldots + x_{n-1} x_n^{d-1}$$

of degree $d \geq 3$ with $n \geq 2$ and $1 \leq a < d - 1$, and prove:

Lemma 2. *The monodromy operator h_g^* associated to g is the identity operator if a and d are coprime.*

Proof. We denote with G the Milnor fibre $(g = 1)$ of g in $\mathbf{C}^{n+1}$. We want to find a $\mathbf{C}^*$-action on $\mathbf{C}^{n+1}$ such that G is invariant and the natural geometric monodromy $h_g : G \to G$ is given by "multiplication" $x \mapsto \lambda \bullet x$ (with respect to that action) by some element $\lambda \in \mathbf{C}^*$. Since $\mathbf{C}^*$ is connected, this will imply that h_g is homotopy equivalent to the identity, thus proving the lemma.

As g is homogeneous of degree d, the geometric monodromy takes the same nice form $h_g(x_0, \ldots, x_n) = (\zeta x_0, \ldots, \zeta x_n)$ with $\zeta := \exp(2\pi i/d)$ as above. The $\mathbf{C}^*$-action will be of diagonal form $t \bullet (x_0, \ldots, x_n) = (t^{q_0} x_0, \ldots, t^{q_n} x_n)$ given by a vector $\mathfrak{q} = (q_0, \ldots, q_n)$ of integral weights $q_j = \mathfrak{q}\text{-}\deg(x_j)$. As G has to be invariant under that action, $\mathfrak{q}$ must be so chosen that g is q-homogeneous with $\mathfrak{q}\text{-}\deg(g) = 0$. Hence, we have the condition

$$a q_0 + (d - a) q_1 = q_1 + (d - 1) q_2 = \ldots = q_{n-1} + (d - 1) q_n = 0$$

which is clearly satisfied by taking $q_n = a$, $q_{n-1} = (1-d)a, \ldots, q_1 = (1-d)^{n-1} a$ and $q_0 = (1 - d)^{n-1}(a - d)$. As a and d are coprime by assumption, we can find an integer b with $ab \equiv 1 \mod d$. Since all weights satisfy $q_i \equiv a \mod d$, the element $\lambda := \zeta^b$ has the property that $\lambda^{q_i} = \zeta$, so $\lambda_\bullet(x_0, \dots, x_n) = h_g(x_0, \dots, x_n)$ as claimed at the beginning. •

Remark. Note that the affine hypersurface $(g_{d,a} = 0)$ in $\mathbf{C}^{n+1}$ has a one-dimensional singular locus, namely, the x_0-axis for $a = 1$, and the union of the x_0- and the x_1-axis for $a > 1$. — For *isolated* hypersurface singularities, the monodromy operator is the identity only in the case of an odd-dimensional A_1-singularity, as follows from the results of A'Campo [–: Thme. 2].

(iii) As indicated above, we now add the monomials x_{n+1}^d to the polynomials $g = g_{d,a}$, thus obtaining the homogeneous polynomials $f := f_{d,a}$ that define our hypersurfaces.

Lemma 3. *Denote with* $f := f_{d,a}$ *the polynomial*

$$f_{d,a}(x_0, x_1, \dots, x_n, x_{n+1}) := g_{d,a}(x_0, x_1, \dots, x_n) + x_{n+1}^d$$

and with $V := V_{n,d}^a$ *the hypersurface in* $\mathbf{P}_{n+1}$ *defined by* $(f = 0)$*. Then* V *has isolated singularities. Moreover,* V *is a rational homology-*$\mathbf{P}_n$ *, i.e., we have* $H_\bullet(V_{n,d}^a, \mathbf{Q}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Q})$*, if* a *and* d *are coprime.*

Proof. The singularities of V are easily determined: Denote with $\mathfrak{o}_i$ (for $i = 0, \dots, n+1$) the origin of the standard affine coordinate system $(x_i = 1)$ on $\mathbf{P}_{n+1}$. The affine equation for V at $\mathfrak{o}_0$ is $f_0 = x_1^{d-a} + x_1 x_2^{d-1} + \dots + x_{n-1} x_n^{d-1} + x_{n+1}^d$, so $\mathfrak{o}_0$ is always an isolated singular point. At $\mathfrak{o}_1$, we have the affine equation $f_1 = x_0^a + x_2^{d-1} + x_2 x_3^{d-1} + \dots + x_{n-1} x_n^{d-1} + x_{n+1}^d$, so $\mathfrak{o}_1$ is a singular point if (and only if) $a > 1$. It is easy to see that there are no other singularities.

To show that $V_{n,d}^a$ is a rational homology- $\mathbf{P}_n$ if $\gcd(a,d) = 1$ holds, it suffices to verify that all eigenvalues of the monodromy operator h_f^* are different from 1 (Lemma 1, (β)). To that end, we can apply results of Oka and use Lemma 2, as f is the sum $g \oplus r$ of the parts $g(x_0, \dots, x_n)$ and $r(x_{n+1}) := x_{n+1}^d$ (with distinct variables). By [$-_1$: Thm.1, Cor.2], the Milnor fibre F of f is homotopy equivalent to the join $G * R$ of the Milnor fibres of g and r, and the monodromy operator h_f^* on $\tilde{H}^{\bullet+1}(F, \mathbf{Q}) \cong (\tilde{H}^\bullet(G, \mathbf{Q}) \otimes \tilde{H}^\bullet(R, \mathbf{Q}))$ is induced from the join of the geometric monodromies. As R is zero-dimensional, that implies the equality $h_f^* = h_g^* \otimes h_r^*$ (which would follow from the Thom-Sebastiani theorem if g had isolated singularities). Since h_g^* is the identity on $\tilde{H}^\bullet(G, \mathbf{Q})$ by Lemma 2, whereas all eigenvalues of h_r^* on $\tilde{H}^\bullet(R, \mathbf{Q})$ are different from 1, we are done. •

Remark. As is well known (cf. the remark following Theorem 1), the rational cohomology ring of any n-dimensional projective variety V contains a graded subring isomorphic to $H^*(\mathbf{P}_n, \mathbf{Q})$. Hence, if V has the rational homology of $\mathbf{P}_n$, then it even has the same rational cohomology ring as $\mathbf{P}_n$, so in particular, rational Poincaré duality holds. Thus, if V has isolated singularities, it follows from L. Kaup's long exact Poincaré duality sequence (see the introduction in [$-_1$]) that V is a rational homology manifold, i.e., all the singularities of V have links that are rational homology spheres (see also [Di_2: Cor.(2.9)]). Moreover, a rational homology-$\mathbf{P}_n$ has the same rational homotopy type as $\mathbf{P}_n$, as the latter is determined by the rational cohomology ring (see Babenko [–: §2]).

(iv) In order to show that we can actually obtain *integral* homology-$\mathbf{P}_n$'s among these varieties $V^a_{n,d}$, we now consider an arbitrary hypersurface $V \subset \mathbf{P}_{n+1}$ with isolated singularities. It is a well known consequence of Lefschetz type theorems, duality theory, and universal coefficient formulae that $H_j(V, \mathbf{Z})$ and $H_j(\mathbf{P}_n, \mathbf{Z})$ are isomorphic except for the two middle dimensions $j = n$ and $n+1$ (see, e.g., [STH: 5.2.6, 5.2.11]). Hence, we only have to discuss these two exceptional cases.

To that end, we use results of [Di_1]. By [–: Thm. 2.1] (or [STH: 5.4.3]), there are isomorphisms

$$H_j(V, \mathbf{Z}) \cong H_j(\mathbf{P}_n, \mathbf{Z}) \oplus \begin{cases} \operatorname{coker} \varphi_V & \text{for } j = n \ , \\ \ker \varphi_V & \text{for } j = n+1 \ , \end{cases}$$

where $\varphi_V : \bigoplus_i L_i \to \overline{L}$ denotes a natural homomorphism of Milnor lattices associated to $V \subset \mathbf{P}_{n+1}$. (Recall that the Milnor lattice of an isolated n-dimensional hypersurface singularity is the reduced integral homology $\tilde{H}_n(F, \mathbf{Z})$ of the corresponding Milnor fibre F, endowed with the intersection form. The lattice—i.e., the form—is symmetric if the dimension n is even, and skew-symmetric if n is odd.) The source of φ_V is the (orthogonal) direct sum of the Milnor lattices L_i at the singular points of V. The target is the *reduced* Milnor lattice $\overline{L} := L/(\mathrm{Rad}\ L)$ associated to the singularity at the origin of the affine cone corresponding to a generic (and thus smooth) hyperplane section of V. Hence, by deformation, that lattice $\overline{L} = \overline{L}_{n,d}$ depends only on n and on the degree d of V. In particular, the middle homology $H_{n,d} := H_n(\tilde{V}, \mathbf{Z})$ of a *smooth* hypersurface $\tilde{V} = \tilde{V}_{n,d} \subset \mathbf{P}_{n+1}$ is isomorphic (as group) to $\overline{L}_{n,d} \oplus H_n(\mathbf{P}_n, \mathbf{Z})$. In fact, $\overline{L}_{n,d}$ can be identified with the sublattice $\ker(j_* : H_n(\tilde{V}_{n,d}) \to H_n(\mathbf{P}_{n+1})) \subset H_{n,d}$ of "vanishing cycles"; cf. [KuWo: 6]. As a consequence, if the dimension n is odd, then $\overline{L}_{n,d}$ and $H_{n,d}$ agree, so $\overline{L}$ is unimodular. On the other hand, if n is even, then $\overline{L}$ is the orthogonal complement $h^\perp$ to the corresponding iterated hyperplane section class h, so it has determinant $\pm d$ (see [Di_1: Cor. 1.4, 1.5, and Rem. 2.4]). As a side-remark, we mention that the lattice homomorphism φ_V also determines the cohomology ring structure of V (see [–: Prop. 6.1]).

To apply the result, we note that $H_{n+1}(V, \mathbf{Z})$ is always torsion free (see [–: Cor. 2.3]). Hence, if V is a rational homology-$\mathbf{P}_n$, then φ_V is a monomorphism—so all L_i are nondegenerate—and its cokernel is a finite torsion group whose order satisfies $|\operatorname{coker}\varphi_V|^2 = \pm(\prod_i \det L_i) / \det \overline{L}$. This yields the following criterion:

Lemma 4. *Let $V \subset \mathbf{P}_{n+1}$ be a rational homology-$\mathbf{P}_n$ of degree d with isolated singularities. The following conditions are equivalent:*
(α) V is an integral homology-$\mathbf{P}_n$;
(β) the cokernel of the lattice homomorphism φ_V is trivial;
(γ) $\prod_i \det L_i = \pm \det \overline{L} = \begin{cases} \pm d & \textit{if } n \textit{ is even,} \\ \pm 1 & \textit{if } n \textit{ is odd.} \end{cases}$ •

We can replace condition (γ) by an equivalent one, using the relation between the intersection form, the monodromy operator, and the "variation operator" (or Seifert form) of an isolated hypersurface singularity defined by a polynomial equation ($p = 0$) (see, e.g., Lamotke's paper [–: §6, Hauptsatz] or the book [SDM II: 2.5] for that relation): The determinant of the Milnor lattice satisfies $\det L_p = \pm\Delta_p$, where $\Delta_p := \Delta_p(1)$ is the value at $t = 1$ of the characteristic polynomial $\Delta_p(t) := \det(t \cdot I - h_p^*)$ of the monodromy operator. Hence, (γ) is equivalent to the condition

(γ') $\prod_i \Delta_i = \begin{cases} \pm d & \textit{if } n \textit{ is even,} \\ \pm 1 & \textit{if } n \textit{ is odd} \end{cases}$

(where Δ_i corresponds to L_i, of course).

(v) The previous lemma has interesting consequences, so we briefly digress to discuss some of them. By one of the classical results in Milnor's book [SPCH: Thm. 8.5], an isolated hypersurface singularity link $K := (p = 0) \cap S_\varepsilon^{2n+1}$ is an integral homology $(2n-1)$-sphere if (and only if) $\Delta_p = \pm 1$. In (complex) dimensions $n \geq 3$, it follows from the generalized Poincaré conjecture that such an integral homology $(2n-1)$-sphere actually is a topological sphere, so the corresponding singularity is a topological manifold point. Moreover, the link K always bounds a parallelizable manifold (see [–: 5.1, 6.1]). Hence, such a singularity link with $\Delta_p = \pm 1$ is h-cobordant to the standard sphere S^{2n-1} if bP_{2n} (the group of h-cobordism classes of $(2n-1)$-dimensional homology spheres that bound a parallelizable $2n$-manifold) is trivial, and that is known to be true for $n = 3$, 7, 15, and 31 (see [KeMi: p. 512] and [HiMa: 10.4, pp. 74/75] for the orders of bP_{2n} with $n \leq 10$, and [Bro: Cor. 2] and [BaJoMa] for $bP_{30} = \{S^{29}\}$ and $bP_{62} = \{S^{61}\}$.) In that case, the smooth structure on the punctured ε-neighbourhood of the singularity can be naturally extended to the singular point (use the h-cobordism to the standard sphere and the fact that a punctured ε-neighbourhood of an isolated singularity is diffeomorphic