1
Introduction

1.1 What are exact solutions, and why study them?

The theories of modern physics generally involve a mathematical model, defined by a certain set of differential equations, and supplemented by a set of rules for translating the mathematical results into meaningful statements about the physical world. In the case of theories of gravitation, it is generally accepted that the most successful is Einstein's theory of general relativity. Here the differential equations consist of purely geometric requirements imposed by the idea that space and time can be represented by a Riemannian (Lorentzian) manifold, together with the description of the interaction of matter and gravitation contained in Einstein's famous field equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = \kappa_0 T_{ab}. \tag{1.1}$$

(The full definitions of the quantities used here appear later in the book.) This book will be concerned only with Einstein's theory. We do not, of course, set out to discuss all aspects of general relativity. For the basic problem of understanding the fundamental concepts we refer the reader to other texts.

For any physical theory, there is first the purely mathematical problem of analysing, as far as possible, the set of differential equations and of finding as many exact solutions, or as complete a general solution, as possible. Next comes the mathematical and physical interpretation of the solutions thus obtained; in the case of general relativity this requires global analysis and topological methods rather than just the purely local solution of the differential equations. In the case of gravity theories, because they deal with the most universal of physical interactions, one has an additional class of problems concerning the influence of the gravitational field on other fields and matter; these are often studied by working within a fixed gravitational field, usually an exact solution.

This book deals primarily with the solutions of the Einstein equations, (1.1), and only tangentially with the other subjects. The strongest reason for excluding the omitted topics is that each would fill (and some do fill) another book; we do, of course, give some references to the relevant literature. Unfortunately, one cannot say that the study of exact solutions has always maintained good contact with work on more directly physical problems. Back in 1975, Kinnersley wrote "Most of the known exact solutions describe situations which are frankly unphysical, and these do have a tendency to distract attention from the more useful ones. But the situation is also partially the fault of those of us who work in this field. We toss in null currents, macroscopic neutrino fields and tachyons for the sake of greater 'generality'; we seem to take delight at the invention of confusing anti-intuitive notation; and when all is done we leave our newborn metric wobbling on its vierbein without any visible means of interpretation." Not much has changed since then.

In defence of work on exact solutions, it may be pointed out that certain solutions have played very important roles in the discussion of physical problems. Obvious examples are the Schwarzschild and Kerr solutions for black holes, the Friedmann solutions for cosmology, and the plane wave solutions which resolved some of the controversies about the existence of gravitational radiation. It should also be noted that because general relativity is a highly non-linear theory, it is not always easy to understand what qualitative features solutions might possess, and here the exact solutions, including many such as the Taub–NUT solutions which may be thought unphysical, have proved an invaluable guide. Though the fact is not always appreciated, the non-linearities also mean that perturbation schemes in general relativity can run into hidden dangers (see e.g. Ehlers *et al.* (1976)). Exact solutions which can be compared with approximate or numerical results are very useful in checking the validity of approximation techniques and programs, see Centrella *et al.* (1986).

In addition to the above reasons for devoting this book to the classification and construction of exact solutions, one may note that although much is known, it is often not generally known, because of the plethora of journals, languages and mathematical notations in which it has appeared. We hope that one beneficial effect of our efforts will be to save colleagues from wasting their time rediscovering known results; in particular we hope our attempt to characterize the known solutions invariantly will help readers to identify any new examples that arise.

One surprise for the reader may lie in the enormous number of known exact solutions. Those who do not work in the field often suppose that the intractability of the full Einstein equations means that very few solutions are known. In a certain sense this is true: we know relatively few exact solutions for real physical problems. In most solutions, for example, there is no complete description of the relation of the field to sources. Problems which are without an exact solution include the two-body problem, the realistic description of our inhomogeneous universe, the gravitational field of a stationary rotating star and the generation and propagation of gravitational radiation from a realistic bounded source. There are, on the other hand, some problems where the known exact solutions may be the unique answer, for instance, the Kerr and Schwarzschild solutions for the final collapsed state of massive bodies.

Any metric whatsoever is a 'solution' of (1.1) if no restriction is imposed on the energy-momentum tensor, since (1.1) then becomes just a definition of T_{ab}; so we must first make some assumptions about T_{ab}. Beyond this we may proceed, for example, by imposing symmetry conditions on the metric, by restricting the algebraic structure of the Riemann tensor, by adding field equations for the matter variables or by imposing initial and boundary conditions. The exact solutions known have all been obtained by making some such restrictions. We have used the term 'exact solution' without a definition, and we do not intend to provide one. Clearly a metric would be called an exact solution if its components could be given, in suitable coordinates, in terms of the well-known analytic functions (polynomials, trigonometric funstions, hyperbolic functions and so on). It is then hard to find grounds for excluding functions defined only by (linear) differential equations. Thus 'exact solution' has a less clear meaning than one might like, although it conveys the impression that in some sense the properties of the metric are fully known; no generally-agreed precise definition exists. We have proceeded rather on the basis that what we chose to include was, by definition, an exact solution.

1.2 The development of the subject

In the first few years (or decades) of research in general relativity, only a rather small number of exact solutions were discussed. These mostly arose from highly idealized physical problems, and had very high symmetry. As examples, one may cite the well-known spherically-symmetric solutions of Schwarzschild, Reissner and Nordström, Tolman and Friedmann (this last using the spatially homogeneous metric form now associated with the names of Robertson and Walker), the axisymmetric static electromagnetic and vacuum solutions of Weyl, and the plane wave metrics. Although such a limited range of solutions was studied, we must, in fairness, point out that it includes nearly all the exact solutions which are of importance in physical applications: perhaps the only one of comparable importance which was discovered after World War II is the Kerr solution.

In the early period there were comparatively few people actively working on general relativity, and it seems to us that the general belief at that time was that exact solutions would be of little importance, except perhaps as cosmological and stellar models, because of the extreme weakness of the relativistic corrections to Newtonian gravity. Of course, a wide variety of physical problems were attacked, but in a large number of cases they were treated only by some approximation scheme, especially the weak-field, slow-motion approximation.

Moreover, many of the techniques now in common use were either unknown or at least unknown to most relativists. The first to become popular was the use of groups of motions, especially in the construction of cosmologies more general than Friedmann's. The next, which was in part motivated by the study of gravitational radiation, was the algebraic classification of the Weyl tensor into Petrov types and the understanding of the properties of algebraically special metrics. Both these developments led in a natural way to the use of invariantly-defined tetrad bases, rather than coordinate components. The null tetrad methods, and some ideas from the theory of group representations and algebraic geometry, gave rise to the spinor techniques, and equivalent methods, now usually employed in the form given by Newman and Penrose. The most recent of these major developments was the advent of the generating techniques, which were just being developed at the time of our first edition (Kramer *et al.* 1980), and which we now describe fully.

Using these methods, it was possible to obtain many new solutions, and this growth is still continuing.

1.3 The contents and arrangement of this book

Naturally, we begin by introducing differential geometry (Chapter 2) and Riemannian geometry (Chapter 3). We do not provide a formal textbook of these subjects; our aim is to give just the notation, computational methods and (usually without proof) standard results we need for later chapters. After this point, the way ahead becomes more debatable.

There are (at least) four schemes for classification of the known exact solutions which could be regarded as having more or less equal importance; these four are the algebraic classification of conformal curvature (Petrov types), the algebraic classification of the Ricci tensor (Plebański or Segre types) and the physical characterization of the energy-momentum tensor, the existence and structure of preferred vector fields, and the groups of symmetry 'admitted by' (i.e. which exist for) the metric (isometries and homotheties). We have devoted a chapter (respectively, Chapters 4, 5, 6 and 8) to each of these, introducing the terminology and methods used later and some general theorems. Among these chapters we have interpolated one (Chapter 7) which gives the Newman–Penrose formalism; its position is due to the fact that this formalism can be applied immediately to elucidating some of the relationships between the considerations in the preceding three chapters. With more solutions being known, unwitting rediscoveries happened more frequently; so methods of invariant characterization became important which we discuss in Chapter 9. We close Part I with a presentation of the generation methods which became so fruitful in the 1980s. This is again one of the subjects which, ideally, warrants a book of its own and thus we had to be very selective in the choice and manner of the material presented.

The four-dimensional presentation of the solutions which would arise from the classification schemes outlined above may be acceptable to relativists but is impractical for authors. We could have worked through each classification in turn, but this would have been lengthy and repetitive (as it is, the reader will find certain solutions recurring in various disguises). We have therefore chosen to give pride of place to the two schemes which seem to have had the widest use in the discovery and construction of new solutions, namely symmetry groups (Part II of the book) and Petrov types (Part III). The other main classifications have been used in subdividing the various classes of solutions discussed in Parts II and III, and they are covered by the tables in Part V. The application of the generation techniques and some other ways of classifying and constructing exact solutions are presented in Part IV.

The specification of the energy-momentum tensor played a very important role because we decided at an early stage that it would be impossible to provide a comprehensive survey of all energy-momentum tensors that have ever been considered. We therefore restricted ourselves to the following energy-momentum tensors: vacuum, electromagnetic fields, pure radiation, dust and perfect fluids. (The term 'pure radiation' is used here for an energy-momentum tensor representing a situation in which all the energy is transported in one direction with the speed of light: such tensors are also referred to in the literature as null fields, null fluids and null dust.) Combinations of these, and matching of solutions with equal or different energy-momentum tensors (e.g. the Schwarzschild vacuoli in a Friedmann universe) are in general not considered, and the cosmological constant Λ, although sometimes introduced, is not treated systematically throughout.

These limitations on the scope of our work may be disappointing to some, especially those working on solutions containing charged perfect fluids, scalar, Dirac and neutrino fields, or solid elastic bodies. They were made not only because some limits on the task we set ourselves were necessary, but also because most of the known solutions are for the energy-momentum tensors listed and because it is possible to give a fairly full systematic treatment for these cases. One may also note that unless additional field equations for the additional variables are introduced, it is easier to find solutions for more complex energy-momentum tensor forms than for simpler ones: indeed in extreme cases there may be no equations to solve at all, the Einstein equations instead becoming merely definitions of the energy-momentum from a metric ansatz. Ultimately, of course, the choice is a matter of taste.

The arrangement within Part II is outlined more fully in §11.1. Here we remark only that we treated first non-null and then null group orbits (as defined in Chapter 8), arranging each in order of decreasing dimension of the orbit and thereafter (usually) in decreasing order of dimension of the group. Certain special cases of physical or mathematical interest were separated out of this orderly progression and given chapters of their own, for example, spatially-homogeneous cosmologies, spherically-symmetric solutions, colliding plane waves and the inhomogeneous fluid solutions with symmetries. Within each chapter we tried to give first the differential geometric results (i.e. general forms of the metric and curvature) and then the actual solutions for each type of energy-momentum in turn; this arrangement is followed in Parts III and IV also.

In Part III we have given a rather detailed account of the well-developed theory that is available for algebraically special solutions for vacuum, electromagnetic and pure radiation fields. Only a few classes, mostly very special cases, of algebraically special perfect-fluid solutions have been thoroughly discussed in the literature: a short review of these classes is given in Chapter 33. Quite a few of the algebraically special solutions also admit groups of motions. Where this is known (and, as far as we are aware, it has not been systematically studied for all cases), it is of course indicated in the text and in the tables.

Part IV, the last of the parts treating solutions in detail, covers solutions found by the generation techniques developed by various authors since 1980 (although most of these rely on the existence of a group of motions, and in some sense therefore belong in Part II). There are many such techniques in use and they could not all be discussed in full: our choice of what to present in detail and what to mention only as a reference simply reflects our personal tastes and experiences. This part also gives some discussion of the classification of space-times with special vector and tensor fields and solutions found by embedding or the study of metrics with special subspaces.

The weight of material, even with all the limitations described above, made it necessary to omit many proofs and details and give only the necessary references.

1.4 Using this book as a catalogue

This book has not been written simply as a catalogue. Nevertheless, we intended that it should be possible for the book to be used for this purpose. In arranging the information here, we have assumed that a reader who wishes to find (or, at least, search for) a solution will know the original author (if the reader is aware the solution is not new) or know some of its invariant properties.

If the original author[1] is known, the reader should turn to the alphabetically-organized reference list. He or she should then be able to identify the relevant paper(s) of that author, since the titles, and, of course, journals and dates, are given in full. Following each reference is a list of all the places in the book where it is cited.

A reader who knows the (maximal) group of motions can find the relevant chapter in Part II by consulting the contents list or the tables. If the reader knows the Petrov type, he or she can again consult the contents list or the tables by Petrov type; if only the energy-momentum tensor is known, the reader can still consult the relevant tables. If none of this information is known, he or she can turn to Part IV, if one of the special methods described there has been used. If still in doubt, the whole book will have to be read.

If the solution is known (and not accidentally omitted) it will in many cases be given in full, possibly only in the sense of appearing contained in a more general form for a whole class of solutions: some solutions of great complexity or (to us) lesser importance have been given only in the sense of a reference to the literature. Each solution may, of course, be found in a great variety of coordinate forms and be characterized invariantly in several ways. We have tried to eliminate duplications, i.e. to identify those solutions that are really the same but appear in the literature as separate, and we give cross-references between sections where possible. The solutions are usually given in coordinates adapted to some invariant properties, and it should therefore be feasible (if non-trivial) for the reader to transform to any other coordinate system he or she has discovered (see also Chapter 9). The many solutions obtained by generating techniques are for the most part only tabulated and not given explicitly,

[1] There is a potential problem here if the paper known to the reader is an unwitting re-discovery, since for brevity we do not cite such works.

since it is in principle possible to generate infinitely many such solutions by complicated but direct calculations.

Solutions that are neither given nor quoted are either unknown to us or accidentally omitted, and in either case the authors would be interested to hear about them. (We should perhaps note here that not all papers containing frequently-rediscovered solutions have been cited: in such a case only the earliest papers, and those rediscoveries with some special importance, have been given. Moreover, if a general class of solutions is known, rediscoveries of special cases belonging to this class have been mentioned only occasionally. We have also not in general commented, except by omission, on papers where we detected errors, though in a few cases where a paper contains some correct and some wrong results we have indicated that.)

We have checked most of the solutions given in the book. This was done by machine and by hand, but sometimes we may have simply repeated the authors' errors. It is not explicitly stated where we did not check solutions.

In addition to references within the text, cited by author and year, we have sometimes put at the ends of sections some references to parallel methods, or to generalizations, or to applications. We would draw the reader's attention to some books of similar character which have appeared since the first edition of this book was published and which complement and supplement this one. Krasiński (1997) has extensively surveyed those solutions which contain as special cases the Robertson–Walker cosmologies (for which see Chapter 14), without the restrictions on energy-momentum content which we impose. Griffiths (1991) gives an extensive study of the colliding wave solutions discussed here in Chapter 25, Wainwright and Ellis (1997) similarly discusses spatially-homogeneous and some other cosmologies (see Chapters 14 and 23), Bičák (2000) discusses selected exact solutions and their history, and Belinski and Verdaguer (2001) reviews solitonic solutions obtainable by the methods of Chapter 34, especially §34.4: these books deal with physical and interpretational issues for which we do not have space.

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Part I
General methods

2
Differential geometry without a metric

2.1 Introduction

The concept of a tensor is often based on the law of transformation of the components under coordinate transformations, so that coordinates are explicitly used from the beginning. This calculus provides adequate methods for many situations, but other techniques are sometimes more effective. In the modern literature on exact solutions coordinatefree geometric concepts, such as forms and exterior differentiation, are frequently used: the underlying mathematical structure often becomes more evident when expressed in coordinatefree terms.

Hence this chapter will present a brief survey of some of the basic ideas of differential geometry. Most of these are independent of the introduction of a metric, although, of course, this is of fundamental importance in the space-times of general relativity; the discussion of manifolds with metrics will therefore be deferred until the next chapter. Here we shall introduce vectors, tensors of arbitrary rank, p-forms, exterior differentiation and Lie differentiation, all of which follow naturally from the definition of a differentiable manifold. We then consider an additional structure, a covariant derivative, and its associated curvature; even this does not necessarily involve a metric. The absence of any metric will, however, mean that it will not be possible to convert 1-forms to vectors, or vice versa.

Since we are primarily concerned with specific applications, we shall emphasize the rules of manipulation and calculation in differential geometry. We do not attempt to provide a substitute for standard texts on the subject, e.g. Eisenhart (1927), Schouten (1954), Flanders (1963), Sternberg (1964), Kobayashi and Nomizu (1969), Schutz (1980), Nakahara (1990) and Choquet-Bruhat *et al.* (1991) to which the reader is referred for fuller information and for the proofs of many of the theorems. Useful introductions can also be found in many modern texts on relativity.

For the benefit of those familiar with the traditional approach to tensor calculus, certain formulae are displayed both in coordinatefree form and in the usual component formalism.

2.2 Differentiable manifolds

Differentiable manifolds are the most basic structures in differential geometry. Intuitively, an (n-dimensional) manifold is a space $\mathcal{M}$ such that any point $p \in \mathcal{M}$ has a neighbourhood $\mathcal{U} \subset \mathcal{M}$ which is homeomorphic to the interior of the (n-dimensional) unit ball. To give a mathematically precise definition of a differentiable manifold we need to introduce some additional terminology.

A *chart* $(\mathcal{U}, \Phi)$ in $\mathcal{M}$ consists of a subset $\mathcal{U}$ of $\mathcal{M}$ together with a one-to-one map Φ from $\mathcal{U}$ onto the n-dimensional Euclidean space E^n or an open subset of E^n; Φ assigns to every point $p \in \mathcal{U}$ an n-tuple of real variables, the *local coordinates* $(x^1, \dots, x^n)$. As an aid in later calculations, we shall sometimes use pairs of complex conjugate coordinates instead of pairs of real coordinates, but we shall not consider generalizations to complex manifolds (for which see e.g. Flaherty (1980) and Penrose and Rindler (1984, 1986)).

Two charts $(\mathcal{U}, \Phi)$, $(\mathcal{U}', \Phi')$ are said to be *compatible* if the combined map $\Phi' \circ \Phi^{-1}$ on the image $\Phi(\mathcal{U} \cup \mathcal{U}')$ of the overlap of $\mathcal{U}$ and $\mathcal{U}'$ is a homeomorphism (i.e. continuous, one-to-one, and having a continuous inverse): see Fig. 2.1.

An *atlas* on $\mathcal{M}$ is a collection of compatible charts $(\mathcal{U}_\alpha, \Phi_\alpha)$ such that every point of $\mathcal{M}$ lies in at least one chart neighbourhood $\mathcal{U}_\alpha$. In most cases, it is impossible to cover the manifold with a single chart (an example which cannot be so covered is the n-dimensional sphere, $n > 0$).

An n-dimensional (topological) *manifold* consists of a space $\mathcal{M}$ together with an atlas on $\mathcal{M}$. It is a (C^k or analytic) *differentiable manifold* $\mathcal{M}$ if the maps $\Phi^{'} \circ \Phi^{-1}$ relating different charts are not just continuous but differentiable (respectively, C^k or analytic). Then the coordinates are related by n differentiable (C^k, analytic) functions, with non-vanishing