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Potential Theory in the Complex Plane

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521461207

First published 1995
First paperback edition 2003

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Ransford, Thomas
Potential theory in the complex plane / Thomas Ransford
p. cm. - (London Mathematical Society student texts; 28)
Includes bibliographical references and index
ISBN 0 521 46120 0. - ISBN 0 521 46654 7 (pbk.)
1. Potential theory (Mathematics) 2. Functions of complex variables
I. Title. II. Series.
QA404.7.R36 1995
515.9-dc20 94-38846 CIP

ISBN 978-0-521-46120-7 hardback
ISBN 978-0-521-46654-7 paperback

Transferred to digital printing 2008

Cambridge University Press
978-0-521-46120-7 - Potential Theory in the Complex Plane
Thomas Ransford
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Preface

When first learning potential theory, as a new graduate student, I experienced some difficulty with the literature then available. The choice lay between several excellent but encyclopaedic treatises on the subject, from which it was hard work to extract what was needed, and several equally excellent books on complex variable, each containing a useful chapter on potential theory, but which did not go nearly far enough. This book is an attempt to bridge that gap—indeed it was consciously written as the book that I should have liked to read all those years ago.

Potential theory is the name given to the broad field of analysis encompassing such topics as harmonic and subharmonic functions, the Dirichlet problem, harmonic measure, Green's functions, potentials and capacity. It can be developed in many contexts, ranging from classical potential theory in $\mathbb{R}^n$ and pluripotential theory in $\mathbb{C}^n$ to axiomatic theories in very general spaces. In between there are versions relating to Riemann surfaces and other manifolds, uniform algebras and analytic multifunctions, to say nothing of the connections with Brownian motion and other stochastic processes. However, there is one case which is common to them all: potential theory in the plane. As it contains all the essential ingredients of the subject, yet is relatively easy and quick to treat, it seems to me to be well worth mastering first. This is the subject of the book.

There is also a further goal, hinted at by the use of the word 'complex' in the title. It is to emphasize the very close connection between potential theory and complex analysis. This works both ways. In one direction, the techniques of complex analysis, particularly conformal mapping, can be used to speed up and simplify proofs of some of the results in potential theory. Going the other way, these same theorems in potential theory have a multitude of applications in complex analysis. Examples of the latter are scattered liberally throughout the text, including, for example: Picard's theorem, the Phragmén–Lindelöf principle, the Radó–Stout theorem, Lindelöf's theorem on asymptotic values, the Riemann mapping theorem (including continuity at the boundary), the Koebe one-quarter theorem, Hilbert's lemniscate theorem, and the sharp quantitative form of Runge's theorem.

Chapters 1–5 cover the basic theory. They are pitched at a level suitable for a first-year graduate student. Thus they presuppose a knowledge of elementary complex analysis (Cauchy's theorem, Cauchy's integral formula, the maximum modulus principle, the identity principle and Taylor's theorem), and also of basic measure theory (Lebesgue measure, the monotone convergence theorem, the dominated convergence theorem, Fatou's lemma and Fubini's theorem). On the other hand, it is not assumed that the reader has previously encountered general Borel measures, so there is an appendix on these, including a proof of the Riesz representation theorem. There are exercises at the end of each section, ranging from five-finger problems to thinly-disguised new theorems.

Chapter 6 demands rather more of the reader. It contains a variety of applications of potential theory to other areas of analysis, notably to functional analysis, approximation theory and dynamical systems. Some, like the results on L^p-interpolation and uniform approximation, are classical, while others, such as those relating to spectral theory of Banach algebras and to Hausdorff dimension of Julia sets, are relatively recent. My own interest in potential theory was inspired by such applications, and I have tried to pick a selection that conveys some of the attraction that the subject holds for me. I hope that it also shows that, even though the basic theory is by now fifty years old, there are still interesting new applications to be found.

This book is based upon versions of a graduate course that I gave at Leeds, Cambridge and Brown Universities, and I should like to thank all those who attended, students and faculty, for their invaluable questions and comments. I owe a particular debt of gratitude to Andy Browder, Brian Cole and John Wermer, who encouraged me to write the course up. Also I am grateful to David Tranah and Roger Astley of Cambridge University Press, as well as an anonymous referee, whose suggestions helped shape the book. And finally, I wish to say a big thank-you to my wife Line: for tolerating all my moaning and groaning about the text, for lending me her slow but trusty Mac SE on which most of it was typed, and for producing a new 'theorem', Julian Vincent, just before its completion. I dedicate the book to her.

Cap Rouge, Québec
June 1994

T.J.R.