A Word about Notation

Much of the action in this book takes on a domain (i.e. a non-empty, connected, open set) in the complex plane $\mathbb{C}$ or the Riemann sphere $\mathbb{C}_\infty$. Though working in the latter may seem out of keeping with the title of the book, it is often beneficial to do so even if one's primary interest is in plane domains, not least because closed subsets of the sphere are always compact. For this reason we shall adopt the convention that, given a subset S of $\mathbb{C}$ (or $\mathbb{C}_\infty$), its closure $\overline{S}$ and its boundary ∂S will *always* be taken relative to $\mathbb{C}_\infty$. In particular, it is important to remember that if S is an unbounded subset of $\mathbb{C}$ then $\infty \in \partial S$; reminders about this will be issued from time to time.

The other piece of notation that will be needed straightaway is a symbol for discs. Given $w \in \mathbb{C}$ and $\rho > 0$, we write

$$\begin{aligned} \Delta(w,\rho) &:= \{z \in \mathbb{C} : |z-w| < \rho\}, \\ \overline{\Delta}(w,\rho) &:= \{z \in \mathbb{C} : |z-w| \le \rho\}. \end{aligned}$$

Although Δ is also the standard symbol for the Laplacian, namely

$$\Delta f := f_{xx} + f_{yy},$$

no confusion should arise since one Δ is followed by a bracket and the other by a function!

All the remaining notation will be introduced as it is needed, and summarized in a glossary at the end of the book. We are now ready to begin.

Chapter 1

Harmonic Functions

1.1 Harmonic and Holomorphic Functions

Harmonic functions, namely solutions of Laplace's equation, exhibit many properties reminiscent of those of holomorphic functions. In fact, when working in the plane, as we shall, there is a direct connection between the two classes. We shall unashamedly exploit this to accelerate the initial development of harmonic functions, under the assumption that we already know something about holomorphic ones. Later, potential theory will repay its debt to complex analysis in the form of many beautiful applications.

We begin with the formal definition.

Definition 1.1.1 Let U be an open subset of $\mathbb{C}$. A function $h: U \to \mathbb{R}$ is called *harmonic* if $h \in C^2(U)$ and $\Delta h = 0$ on U.

The following basic result not only furnishes numerous examples of harmonic functions, but also provides a useful tool in deriving their elementary properties from those of holomorphic functions.

Theorem 1.1.2 *Let D be a domain in $\mathbb{C}$.*

(a) *If f is holomorphic on D and $h = \operatorname{Re} f$, then h is harmonic on D.*

(b) *If h is harmonic on D, and if D is simply connected, then $h = \operatorname{Re} f$ for some f holomorphic on D. Moreover f is unique up to adding a constant.*

Proof. (a) Writing $f = h + ik$, the Cauchy–Riemann equations give that $h_x = k_y$ and $h_y = -k_x$. Therefore

$$\Delta h = h_{xx} + h_{yy} = k_{yx} - k_{xy} = 0.$$

(b) If $h = \operatorname{Re} f$ for some holomorphic function f, say $f = h + ik$, then

$$f' = h_x + ik_x = h_x - ih_y. \tag{1.1}$$

Thus, if f exists, then f' is completely determined by h, and hence f is unique up to adding a constant.

Equation (1.1) also suggests how we might construct such a function f. Define $g: D \to \mathbb{C}$ by

$$g = h_x - ih_y.$$

Then $g \in C^1(D)$ and g satisfies the Cauchy–Riemann equations because

$$h_{xx} = -h_{yy} \quad \text{and} \quad h_{xy} = h_{yx}.$$

Therefore g is holomorphic on D. Fix $z_0 \in D$, and define $f: D \to \mathbb{C}$ by

$$f(z) = h(z_0) + \int_{z_0}^{z} g(w)\,dw,$$

the integral being taken over any path in D from z_0 to z. As D is simply connected, Cauchy's theorem ensures that the integral is independent of the particular path chosen. Then f is holomorphic on D and $f' = g = h_x - ih_y$. Writing $\widetilde{h} = \operatorname{Re} f$, we have

$$\widetilde{h}_x - i\widetilde{h}_y = f' = h_x - ih_y,$$

so that $(\widetilde{h} - h)_x \equiv 0$ and $(\widetilde{h} - h)_y \equiv 0$. It follows that $\widetilde{h} - h$ is constant on D, and putting $z = z_0$ shows that the constant is zero. Thus indeed $h = \operatorname{Re} f$. □

As a consequence, we obtain a useful result about holomorphic logarithms.

Corollary 1.1.3 *Let f be holomorphic and non-zero on a simply connected domain D in $\mathbb{C}$. Then there exists a holomorphic function g on D such that $f = e^g$.*

Proof. Put $h = \log|f|$ on D. Because h is locally the real part of a holomorphic function, namely a branch of $\log f$, it is harmonic by Theorem 1.1.2 (a). By Theorem 1.1.2 (b) there exists g holomorphic on D such that $h = \operatorname{Re} g$ there, or in other words, $|fe^{-g}| = 1$ on D. By the maximum principle for holomorphic functions, fe^{-g} is a constant C. Adding a suitable constant to g, we can suppose that $C = 1$, and so $f = e^g$. □

Corollary 1.1.3 (and, by implication, Theorem 1.1.2 (b)) may fail if D is not simply connected. For example, the function $f(z) = z$ is holomorphic and non-zero on the domain $D = \mathbb{C} \setminus \{0\}$; but there is no holomorphic function g such that $z = e^{g(z)}$ on this domain, for such a g would satisfy $g'(z) = 1/z$, and this would then imply that

$$0 = \int_{|z|=1} g'(z)\,dz = \int_{|z|=1} \frac{1}{z}\,dz = 2\pi i,$$

which is obviously false.

However, since discs are simply connected, every harmonic function is at least locally the real part of some holomorphic function. This has the following immediate consequences.

Corollary 1.1.4 *If h is a harmonic function on an open subset U of $\mathbb{C}$, then $h \in C^\infty(U)$.* □

Corollary 1.1.5 *If $f: U_1 \to U_2$ is a holomorphic map between open subsets U_1, U_2 of $\mathbb{C}$, and if h is harmonic on U_2, then $h \circ f$ is harmonic on U_1.* □

This result allows us to extend the notion of harmonicity to the Riemann sphere. Given a function h defined on an open neighbourhood U of ∞, we say h is harmonic on U if $h \circ \phi^{-1}$ is harmonic on $\phi(U)$, where ϕ is a conformal mapping of U onto an open subset of $\mathbb{C}$. It does not matter which map ϕ is chosen: if ϕ_1 and ϕ_2 are two such choices, then $(h \circ \phi_1^{-1}) = (h \circ \phi_2^{-1}) \circ f$, where $f = \phi_2 \circ \phi_1^{-1}$, so by Corollary 1.1.5 $h \circ \phi_1^{-1}$ is harmonic on $\phi_1(U)$ if and only if $h \circ \phi_2^{-1}$ is harmonic on $\phi_2(U)$.

Another simple consequence of Theorem 1.1.2 will be of great importance later.

Theorem 1.1.6 (Mean-Value Property) *Let h be a function harmonic on an open neighbourhood of the disc $\overline{\Delta}(w, \rho)$. Then*

$$h(w) = \frac{1}{2\pi} \int_0^{2\pi} h(w + \rho e^{i\theta})\,d\theta.$$

Proof. Choose $\rho' > \rho$ so that h is harmonic on $\Delta(w, \rho')$. Applying Theorem 1.1.2 (b), there exists f holomorphic on $\Delta(w, \rho')$ such that $h = \operatorname{Re} f$ there. By Cauchy's integral formula, we have

$$f(w) = \frac{1}{2\pi i} \int_{|\zeta - w| = \rho} \frac{f(\zeta)}{\zeta - w}\,d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(w + \rho e^{i\theta})\,d\theta.$$

The result follows upon taking real parts of both sides. □

This section ends with two further ways in which harmonic functions behave like holomorphic ones, an identity principle and a maximum principle. We shall deduce the harmonic versions of both these results from their holomorphic counterparts.

Theorem 1.1.7 (Identity Principle) *Let h and k be harmonic functions on a domain D in $\mathbb{C}$. If $h = k$ on a non-empty open subset U of D, then $h = k$ throughout D.*

Proof. We can suppose, without loss of generality, that $k = 0$. Set $g = h_x - ih_y$. Then as in the proof of Theorem 1.1.2, g is holomorphic on D, and also $g = 0$ on U since $h = 0$ there. By the identity principle for holomorphic functions, it follows that $g = 0$ throughout D, and hence that $h_x = 0$ and $h_y = 0$ on D. Therefore h is constant on D, and since $h = 0$ on U, this constant must be zero. □

For holomorphic functions, a stronger form of identity principle holds: namely, if two holomorphic functions agree on a set with a limit point in the domain D, then they agree throughout D. However, this is not the case for harmonic functions. For instance, the functions $h(z) = \operatorname{Re} z$ and $k(z) = 0$ are harmonic on $\mathbb{C}$ and agree on the imaginary axis without being equal on the whole of $\mathbb{C}$.

Theorem 1.1.8 (Maximum Principle) *Let h be a harmonic function on a domain D in $\mathbb{C}$.*

(a) *If h attains a local maximum on D, then h is constant.*

(b) *If h extends continuously to $\overline{D}$ and $h \le 0$ on ∂D, then $h \le 0$ on D.*

This is perhaps a timely moment for a reminder about our convention that all closures and boundaries are taken with respect to $\mathbb{C}_\infty$ rather than $\mathbb{C}$. Indeed Theorem 1.1.8 (b) would otherwise be false: consider, for example, the harmonic function $h(z) = \operatorname{Re} z$ on the domain $D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Proof. (a) Suppose that h attains a local maximum at $w \in D$. Then for some $r > 0$ we have $h \le h(w)$ on $\Delta(w, r)$. By Theorem 1.1.2 (b) there exists a function f holomorphic on $\Delta(w, r)$ such that $h = \operatorname{Re} f$ there. Then $|e^f|$ attains a local maximum at w, so e^f must be constant. Therefore h is constant on $\Delta(w, r)$, and hence on the whole of D by the identity principle.

(b) As $\overline{D}$ is compact, h must attain a maximum at some point $w \in \overline{D}$. If $w \in \partial D$, then $h(w) \le 0$ by assumption, and so $h \le 0$ on D. If $w \in D$, then by part (a) h is constant on D, hence on $\overline{D}$, and so once again $h \le 0$ on D. □

Exercises 1.1

1. Let $h(x+iy) = e^x(x\cos y - y\sin y)$. Show that h is harmonic on $\mathbb{C}$, and find a holomorphic function f on $\mathbb{C}$ such that $h = \operatorname{Re} f$.
2. Let h be a function harmonic on $\{z \in \mathbb{C} : \rho_1 < |z| < \rho_2\}$. Using the fact that $h_x - ih_y$ is holomorphic, prove that there exist unique constants $(a_n)_{n\in\mathbb{Z}}$ and b, with $a_0, b \in \mathbb{R}$, such that
$$h(z) = \operatorname{Re}\left(\sum_{-\infty}^{\infty} a_n z^n\right) + b\log|z| \quad (\rho_1 < |z| < \rho_2).$$
3. Let h, k be functions which are harmonic and non-constant on a domain D. Prove that hk is harmonic if and only if $h+ick$ is holomorphic for some real constant c. [Hint for the 'only if': consider f/g, where $f = h_x - ih_y$ and $g = k_x - ik_y$.]
4. Show that every harmonic function is real-analytic, and use this to give another proof of the identity principle.
5. Show that the only functions harmonic on the whole of $\mathbb{C}_\infty$ are the constants.

1.2 The Dirichlet Problem on the Disc

The Dirichlet problem is to find a harmonic function on a domain with prescribed boundary values. It is one of the great advantages of harmonic functions over holomorphic ones that for 'nice' domains, a solution always exists. This is a powerful tool with many applications.

Here is the formal statement of the problem.

Definition 1.2.1 Let D be a subdomain of $\mathbb{C}$, and let $\phi: \partial D \to \mathbb{R}$ be a continuous function. The *Dirichlet problem* is to find a harmonic function h on D such that $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial D$.

The question of uniqueness is easily settled.

Theorem 1.2.2 (Uniqueness Theorem) *With the notation of Definition 1.2.1, there is at most one solution h to the Dirichlet problem.*

Proof. Suppose that h_1 and h_2 are both solutions. Then $h_1 - h_2$ is harmonic on D, extends continuously to $\overline{D}$, and is zero on ∂D. Applying the maximum principle Theorem 1.1.8 (b) to $\pm(h_1 - h_2)$, we conclude that $h_1 - h_2 = 0$. □

The question of existence of solutions to the Dirichlet problem is rather more delicate and is postponed until Chapter 4. However, there is one important special case that we can solve now, namely when D is a disc. To this end, we make the following definition.

Definition 1.2.3 (a) The *Poisson kernel* $P: \Delta(0,1) \times \partial\Delta(0,1) \to \mathbb{R}$ is defined by

$$P(z,\zeta) := \operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{1-|z|^2}{|\zeta-z|^2} \quad (|z|<1, |\zeta|=1).$$

(b) If $\Delta = \Delta(w,\rho)$ and $\phi: \partial\Delta \to \mathbb{R}$ is a Lebesgue-integrable function, then its *Poisson integral* $P_\Delta\phi: \Delta \to \mathbb{R}$ is defined by

$$P_\Delta\phi(z) := \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z-w}{\rho}, e^{i\theta}\right)\phi(w+\rho e^{i\theta})\,d\theta \quad (z \in \Delta).$$

More explicitly, if $r < \rho$ and $0 \le t < 2\pi$, then

$$P_\Delta\phi(w+re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-t)+r^2}\phi(w+\rho e^{i\theta})\,d\theta.$$

The following result is fundamental.

Theorem 1.2.4 *With the notation of Definition 1.2.3:*

(a) *$P_\Delta\phi$ is harmonic on Δ;*

(b) *if ϕ is continuous at $\zeta_0 \in \partial\Delta$, then $\lim_{z\to\zeta_0} P_\Delta\phi(z) = \phi(\zeta_0)$.*

In particular, if ϕ is continuous on the whole of $\partial\Delta$, then $h := P_\Delta\phi$ solves the Dirichlet problem on Δ.

Proof. (a) Making an affine change of variable if necessary, we can suppose that $w = 0$ and $\rho = 1$, so that $\Delta = \Delta(0,1)$. Then

$$P_\Delta\phi(z) = \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\phi(e^{i\theta})\,d\theta\right) \quad (z \in \Delta),$$

so that $P_\Delta\phi$ is the real part of a holomorphic function of z. Hence it is harmonic on Δ.

Turning to the proof of part (b), it is convenient first to prove a lemma about the Poisson kernel.

Lemma 1.2.5 *The Poisson kernel P satisfies:*

(i) $P(z,\zeta) > 0 \quad (|z| < 1, |\zeta| = 1)$;

(ii) $\frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\,d\theta = 1 \quad (|z| < 1)$;

(iii) $\sup_{|\zeta-\zeta_0|\ge\delta} P(z,\zeta) \to 0$ *as* $z \to \zeta_0 \quad (|\zeta_0| = 1, \delta > 0)$.

Proof. (i) This is clear from the definition of $P(z,\zeta)$.

(ii) Expressing the given integral as a contour integral and using the Cauchy integral formula, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\,d\theta &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\right)\\ &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1}\left(\frac{2}{\zeta-z}-\frac{1}{\zeta}\right)d\zeta\right)\\ &= \mathrm{Re}(2-1) = 1.\end{aligned}$$

(iii) If $|z-\zeta_0| < \delta$ then

$$\sup_{|\zeta-\zeta_0|\ge\delta} P(z,\zeta) \le \frac{1-|z|^2}{(\delta-|\zeta_0-z|)^2},$$

and the result follows easily from this. □

Proof of Theorem 1.2.4 (b). Once again, we may suppose that $\Delta = \Delta(0,1)$. Then using Lemma 1.2.5 (i) and (ii) we have

$$\begin{aligned}|P_\Delta\phi(z) - \phi(\zeta_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})(\phi(e^{i\theta})-\phi(\zeta_0))\,d\theta\right|\\ &\le \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})|\phi(e^{i\theta})-\phi(\zeta_0)|\,d\theta.\end{aligned}$$

Let $\epsilon > 0$. If ϕ is continuous at ζ_0, then there exists $\delta > 0$ such that

$$|\zeta-\zeta_0| < \delta \Rightarrow |\phi(\zeta)-\phi(\zeta_0)| < \epsilon.$$

Hence, using Lemma 1.2.5 (i) and (ii) again, it follows that

$$\frac{1}{2\pi}\int_{|e^{i\theta}-\zeta_0|<\delta} P(z,e^{i\theta})|\phi(e^{i\theta})-\phi(\zeta_0)|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\epsilon\,d\theta = \epsilon.$$

Also, from Lemma 1.2.5 (iii), there exists $\delta' > 0$ such that

$$|z-\zeta_0| < \delta' \Rightarrow \sup_{|\zeta-\zeta_0|\ge\delta} P(z,\zeta) < \epsilon.$$

Hence if $|z-\zeta_0|<\delta'$ then

$$\begin{aligned}\frac{1}{2\pi}\int_{|e^{i\theta}-\zeta_0|\geq\delta} & P(z,e^{i\theta})|\phi(e^{i\theta})-\phi(\zeta_0)|\,d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\epsilon|\phi(e^{i\theta})-\phi(\zeta_0)|\,d\theta \\ &\leq \epsilon\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi(e^{i\theta})|\,d\theta+|\phi(\zeta_0)|\right).\end{aligned}$$

Combining these facts, we deduce that if $|z-\zeta_0|<\delta'$ then

$$|P_\Delta\phi(z)-\phi(\zeta_0)|\leq\epsilon\left(1+\frac{1}{2\pi}\int_0^{2\pi}|\phi(e^{i\theta})|\,d\theta+|\phi(\zeta_0)|\right).$$

This concludes the proof. □

As an immediate consequence of this result, we obtain an analogue of the Cauchy integral formula for harmonic functions.

Corollary 1.2.6 (Poisson Integral Formula) *If h is harmonic on an open neighbourhood of the disc $\overline{\Delta}(w,\rho)$, then for $r<\rho$ and $0\leq t<2\pi$*

$$h(w+re^{it})=\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-t)+r^2}h(w+\rho e^{i\theta})\,d\theta.$$

Proof. Consider the Dirichlet problem on $\Delta:=\Delta(w,\rho)$ with $\phi=h|_{\partial\Delta}$. By Theorem 1.2.4, h and $P_\Delta h$ are both solutions, so by Theorem 1.2.2, $h=P_\Delta h$ on Δ. □

Note that this result is a generalization of the mean-value property, which is just the case $r=0$. It allows us to recapture the values of h everywhere on Δ from knowledge of h on $\partial\Delta$. Exercise 4 gives an analogous formula for f on Δ, where f is the essentially unique holomorphic function such that $h=\operatorname{Re} f$.

The mean-value property actually characterizes harmonic functions. This is proved in the next theorem, which also illustrates well the value of being able to solve the Dirichlet problem.

Theorem 1.2.7 (Converse to Mean-Value Property) *Let $h\colon U\to\mathbb{R}$ be a continuous function on an open subset U of $\mathbb{C}$, and suppose that it possesses the* local mean-value property, *i.e. given $w\in U$, there exists $\rho>0$ such that*

$$h(w)=\frac{1}{2\pi}\int_0^{2\pi}h(w+re^{it})\,dt\quad(0\leq r<\rho).$$

Then h is harmonic on D.

Proof. It is enough to show that h is harmonic on each open disc Δ with $\overline{\Delta} \subset U$. Fix such a Δ, and define $k: \overline{\Delta} \to \mathbb{R}$ by

$$k = \begin{cases} h - P_\Delta h, & \text{on } \Delta, \\ 0, & \text{on } \partial\Delta. \end{cases}$$

Then k is continuous on $\overline{\Delta}$ and has the local mean-value property on Δ. As $\overline{\Delta}$ is compact, k attains a maximum value M at some point of $\overline{\Delta}$. Define

$$A = \{z \in \Delta : k(z) < M\} \quad \text{and} \quad B = \{z \in \Delta : k(z) = M\}.$$

Then A is open, since k is continuous. Also B is open, for if $k(w) = M$, then the local mean-value property forces k to be equal to M on all sufficiently small circles around w. As A and B partition the connected set Δ, either $A = \Delta$, in which case k attains its maximum on $\partial\Delta$ and so $M = 0$, or else $B = \Delta$, in which case $k \equiv M$ and again $M = 0$. Thus $k \leq 0$, and a similar argument shows that $k \geq 0$. Hence $h = P_\Delta h$ on Δ, and since $P_\Delta h$ is harmonic there, so is h. □

Corollary 1.2.8 *If $(h_n)_{n\geq 1}$ is a sequence of harmonic functions on D converging locally uniformly to a function h, then h is also harmonic on D.*

Proof. Combine Theorems 1.1.6 and 1.2.7. □

A useful feature of Theorem 1.2.7 is that one only needs to check that the mean-value property holds *locally* (i.e. the value of ρ can depend upon w). As an application of this, we derive a form of the reflection principle for holomorphic functions.

Theorem 1.2.9 (Reflection Principle) *Let $\Delta = \Delta(0, R)$, and write*

$$\Delta^+ = \{z \in \Delta : \operatorname{Im} z > 0\}, \quad \text{and} \quad I = \{z \in \Delta : \operatorname{Im} z = 0\}.$$

Suppose that f is a holomorphic function on Δ^+ such that $\operatorname{Re} f$ extends continuously to $\Delta^+ \cup I$ with $\operatorname{Re} f = 0$ on I. Then f extends holomorphically to the whole of Δ.

Note that no assumption is made about continuity of $\operatorname{Im} f$ on I—this comes for free.

Proof. Define $h: \Delta \to \mathbb{R}$ by

$$h(z) = \begin{cases} \operatorname{Re} f(z), & z \in \Delta^+, \\ 0, & z \in I, \\ -\operatorname{Re} f(\overline{z}), & \overline{z} \in \Delta^+. \end{cases}$$