ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Semimodular Lattices

Theory and Applications

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK http://www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011-4211, USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1999

Printed in the United States of America

Typeset in 10/13 Times Roman in LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Stern, Manfred.
Semimodular lattices: theory and applications. / Manfred Stern.
p. cm. – (Encyclopedia of mathematics and its applications ;
v. 73)
Includes bibliographical references and index.
I. Title. II. Series.
QA171.5.S743 1999
511.3′3 – dc21 98-44873
CIP

ISBN 0 521 46105 7 hardback

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From Boolean Algebras to Semimodular Lattices

1.1 Sources of Semimodularity

Summary. We briefly indicate some milestones in the general development of lattice theory. In particular, we outline the way leading from Boolean algebras to semimodular lattices. Most of the concepts mentioned in this section will be explained in more detail later. We also give a number of general references and monographs on lattice theory and its history.

Boolean Algebras and Distributive Lattices

Lattice theory evolved in the nineteenth century through the works of George Boole, Charles Saunders Peirce, and Ernst Schröder, and later in the works of Richard Dedekind, Garrett Birkho21 , Oystein Ore, John von Neumann, and others during the first half of the twentieth century. Boole [1847] laid the foundation for the algebras named after him. Since then the more general distributive lattices have been investigated whose natural models are systems of sets. There are many monographs on Boolean algebras and their applications, such as Halmos [1963] and Sikorski [1964]. For the theory of distributive lattices we refer to the books by Grätzer [1971] and Balbes & Dwinger [1974].

Modular Lattices

Dedekind [1900] observed that the additive subgroups of a ring and the normal subgroups of a group form lattices in a natural way (which he called *Dualgruppen*) and that these lattices have a special property, which was later referred to as the *modular law*. Modularity is a consequence of distributivity, and Dedekind's observation gave rise to examples of nondistributive modular lattices.

Lattice theory became established in the 1930s due to the contributions of Garrett Birkhoff, Ore, Menger, von Neumann, Wilcox, and others.

In a series of papers Ore generalized the classical results of Dedekind and investigated decomposition theorems known from algebra in the context of modular lattices (cf. in particular Ore [1935], [1936]). Kurosch [1935] published a note in which he proved a result, independently of Ore and at the same time, that became later known as the Kurosh–Ore theorem. The motivation behind this work was the conjecture that the corresponding result for commutative rings (which was proved by Noether [1921]) could also be proved in the noncommutative case. Kurosh and Ore reduced Noether's theorem to its basic ingredients. However, their work amounted to more than a mere repetition of the former proof in the more general framework of modular lattices. Modularity is the best-known generalization of distributivity. Distributive and modular lattices are dealt with in all books on lattice theory and universal algebra. There are several monographs treating modular lattices within the framework of continuous geometries (von Neumann [1960], Maeda [1958], and Skornjakov [1961]).

Semimodular Lattices

An important source of examples leading to lattices is based on the idea of considering various collections of points, lines, planes, etc., as geometrical "configurations." For example, projective incidence geometries lead to complemented modular lattices: the lattices of flats or closed subspaces of the geometry. However, if one considers affine incidence geometries, the corresponding lattices of flats are no longer modular, although they retain certain important features of complemented modular lattices. These lattices are special instances of so-called geometric lattices. Properties of lattices of this kind were studied by Birkhoff [1933], [1935a]. During the years 1928–35 Menger and his collaborators independently developed ideas that are closely related (see Menger [1936]). Birkhoff's work [1935b] was inspired by the matroid concept introduced by Whitney [1935] in a paper entitled "On the abstract properties of linear dependence." A matroid is a finite set endowed with a closure operator possessing what is now usually called the Steinitz–Mac Lane exchange property. Matroid theory has developed into a rich and flourishing subject. Crapo & Rota [1970a] present in the introduction to their book a survey of the development of matroid theory and geometric lattices. For more details see also Crapo & Rota [1970b] and Kung [1986b]. For a comprehensive account of older and more recent developments in matroid theory, including numerous contributions and historical notes on the relationship with lattice theory and other fields, we refer to the three volumes White [1986], [1987], [1992].

As Garrett Birkhoff stated, the theory of geometric lattices was not foreshadowed in Dedekind's work. Geometric lattices are atomistic lattices of finite length satisfying the *semimodular implication*

(Sm) If $a \wedge b$ is a lower cover of a, then b is a lower cover of $a \vee b$.

We shall call a lattice (of finite length or not) *upper semimodular* or simply *semimodular* if it satisfies the implication (Sm). Birkhoff originally introduced another condition, namely

(Bi) If $a \wedge b$ is a lower cover of a and b, then a and b are lower covers of $a \vee b$.

In lattices of finite length both conditions are equivalent, but in general *Birkhoff's condition* (Bi) is weaker than (Sm). This is why lattices satisfying (Bi) are sometimes called *weakly semimodular*. In our use of the word *semimodular* we have adopted the terminology used by Crawley & Dilworth [1973]. We remark, however, that the notion *semimodular* has also been used to denote other related conditions, some of which will be considered later. The name *semimodular* was coined by Wilcox [1939].

From Dedekind's isomorphism theorem for modular lattices it is immediate that any modular lattice is semimodular. On the other hand, matroids lead to semimodular lattices that are not modular in general. The implication reversed to (Sm) will be denoted by (Sm*). Lattices satisfying (Sm*) are called *lower semimodular* or *dually semimodular*. For lattices of finite length, (Sm) together with (Sm*) yields modularity. In this sense semimodularity is indeed just "one half" of modularity. However, an upper and lower semimodular lattice of infinite length need not be modular. An example is provided by the orthomodular lattice of closed subspaces of an infinite-dimensional Hilbert space.

Conditions Related to Semimodularity

The semimodular implication (Sm) and Birkhoff's condition (Bi) are stated in terms of the covering relation. Hence they only trivially apply to infinite lattices with continuous chains. For example, the lattice of projection operators of a von Neumann algebra has no atoms; it therefore trivially satisfies (Sm) and could formally be cited as an example of a semimodular lattice. However, not much insight is gained from this observation.

Wilcox and Mac Lane were the first to introduce conditions that do not involve coverings and that may be considered as substitutes for the semimodular implication (Sm) in arbitrary lattices.

Wilcox [1938], [1939] showed that affine geometry as developed algebraically by Menger can be axiomatized without the use of points (now usually called atoms). Wilcox's central concept is the symmetry of modular pairs, called *M-symmetry* and briefly denoted by (Ms). This notion came to play a decisive role in nonmodular lattices. Looking back Wilcox (1988, personal communication) wrote:

> The affine geometries seemed a natural place to start. Karl Menger had done some work here, but not in a way that would lend itself to generalization. As I recall my approach which led naturally to the idea of modular pairs, I noted the obvious fact that the failure of an affine geometry (as a lattice) to be modular stems from the presence of parallel pairs, i.e. pairs whose meets

> are “too small”... . Parallel pairs may be thus viewed as non-modular pairs. As to the symmetry of modularity, I noted that in the affine case parallelism is symmetric, i.e. non-modularity of pairs is symmetric, so that modularity of pairs is also symmetric. Hence I began to look for generalizations of affine geometries in which modularity was symmetric.

We have already stated that the lattice of projection operators of a von Neumann algebra is trivially upper semimodular. Topping [1967] proved the highly nontrivial result that this lattice is M-symmetric. We shall return to M-symmetry on several occasions, especially in Chapter 2.

The condition introduced by Mac Lane [1938] and briefly denoted by (Mac) is somewhat more complicated (for details see Section 3.1). In his investigations on “exchange lattices” Mac Lane was led to this condition when looking for “point-free” substitutes for an axiom due to Menger. For more details on the background of these investigations see Mac Lane [1976].

Wilcox’s condition of M-symmetry (Ms) and Mac Lane’s condition (Mac) are both consequences of modularity. On the other hand, neither (Ms) nor (Mac) implies modularity. Also, both (Ms) and (Mac) imply upper semimodularity, but not conversely. Moreover, (Ms) and (Mac) are independent of each other, that is, neither of these conditions implies the other. However, for lattices of finite length, the conditions (Sm), (Bi), (Ms), and (Mac) are all equivalent.

By a *condition related to semimodularity* I mean a condition that is equivalent to upper semimodularity for lattices of finite length. In this sense Birkhoff’s condition (Bi), Wilcox’s condition (Ms) of M-symmetry, and Mac Lane’s condition (Mac) are conditions related to semimodularity.

Figure 1.1 visualizes the interrelationships between the classes of lattices mentioned before. An arrow indicates proper inclusion, that is, if X and Y are classes of lattices, then $\mathsf{X} \to \mathsf{Y}$ means $\mathsf{X} \subset \mathsf{Y}$.

Other conditions related to semimodularity were discovered by Dilworth and Croisot; these conditions will be considered in Section 3.2.

We shall insert inclusion charts in many places. In particular, we shall give inclusion charts refining Figure 1.1. In these inclusion charts arrowheads will sometimes be omitted with the understanding that, where two concepts are connected by an ascending line, the “lower” concept implies the “upper” one.

Local Distributivity and Local Modularity

The early papers by Dilworth [1940], [1941a] were further milestones and important sources of semimodularity. Many of the decomposition theorems in algebra had already been extended to the more general domains of distributive lattices and modular lattices in the 1930s, for example in the above-mentioned works of Ore and Kurosh. Dilworth observed that there are lattices with very simple arithmetical

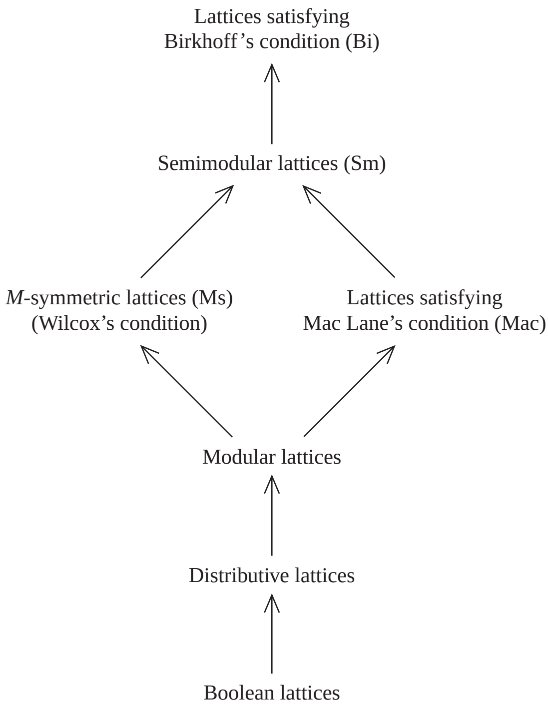


Figure 1.1

properties that come under neither of these classifications. For example, the lattices with unique irreducible meet decompositions that were considered by Dilworth [1940] are upper semimodular. This paper led to the concept of *local distributivity* and marked the origin of a combinatorial structure that is nowadays also referred to as an *antimatroid*. More generally, Dilworth [1941a] investigated *local modularity*; this was the first paper dealing with an extension of the Kurosh–Ore theorem from modular lattices to semimodular lattices.

Notes

Let us first mention some further sources of lattice theory and its history. Mehrtens [1979] gives a detailed account of the development of lattice theory from the very beginnings until about 1940 (for a comprehensive review of Mehrtens's book see Dauben [1986]). Much information on logics as a source of lattice theory can be found in Chapter 2 of Mangione & Bozzi [1993]. An excellent source for tracing the historical development of lattice theory in general and of semimodular lattices in particular is the three editions of Birkhoff's treatise *Lattice Theory* (Birkhoff [1940a], [1948], [1967]). Details on the early history of lattice theory can be found in Birkhoff's General Remark to Chapter I of his *Selected Papers on Algebra and Topology* (see Rota & Oliveira [1987]). Similarly, the *Selected Papers of Robert P. Dilworth* (see Bogart et al. [1990]) provide an invaluable insight into Dilworth's way of thinking and his strategy for solving problems. Dilworth himself wrote background information for the chapters of this book, and his papers are supplemented by comments that trace the influence of his ideas.

For general reference we also list some monographs on lattice theory and related topics in chronological order without claiming completeness: Dubreil-Jacotin et al. [1953], Szász [1963], Barbut & Monjardet [1970], Maeda & Maeda [1970], Blyth & Janowitz [1972], Crawley & Dilworth [1973], Grätzer [1978], [1998], Davey & Priestley [1990], Czédli [1999]. For the role of finite partially ordered sets and lattices in combinatorics see Aigner [1979], Stanley [1986], and Hibi [1992]. For the interplay between lattices and universal algebra we refer to Grätzer [1979] and McKenzie et al. [1987].

Let us also say a few more words about names for concepts. When we use the term *Boolean algebra* we really mean something other than *Boolean lattice* (see Section 1.2). Modular lattices have also been called Dedekind lattices. When we say *semimodular*, we always mean *upper semimodular*. We have already mentioned the dual concept *lower semimodular*. Similarly we shall speak of *upper locally distributive* and its dual *lower locally distributive* as well as of *upper locally modular* and its dual *lower locally modular*. Upper locally distributive lattices are also called join-distributive lattices or locally free lattices (there are also other names for them). Lower locally distributive lattices are also called meet-distributive lattices.

References

Aigner, M. [1979] *Combinatorial Theory*, Springer-Verlag, Berlin.

Balbes, R. and P. Dwinger [1974] *Distributive Lattices*, Univ. of Missouri Press, Columbia, Mo.

Barbut, M. and B. Monjardet [1970] *Ordre et classification*, Vols. 1, 2, Hachette, Paris.

Birkhoff, G. [1933] On the combination of subalgebras, Proc. Camb. Phil. Soc. 29, 441–464.

Birkhoff, G. [1935a] Abstract linear dependence and lattices, Amer. J. Math. 57, 800–804.

Birkhoff, G. [1935b]

Birkhoff, G. [1940a] *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, Vol. 25, New York.

Birkhoff, G. [1948] *Lattice Theory* (2nd edition), Amer. Math. Soc. Colloquium Publications, Vol. 25, New York.

Birkhoff, G. [1967] *Lattice Theory* (3rd edition), Amer. Math. Soc. Colloquium Publications, Vol. 25, Providence, R.I.; reprinted 1984.

Birkhoff, G. [1987] General remark to Chapter I, in: Rota & Oliveira [1987], pp. 1–8.

Blyth, T. S. and M. F. Janowitz [1972] *Residuation Theory*, Pergamon Press, Oxford.

Bogart, K. P., Freese, R., and Kung, J. P. S. (eds.) [1990] *The Dilworth Theorems. Selected Papers of Robert P. Dilworth*, Birkhäuser, Boston.

Boole, G. [1847] *The Mathematical Analysis of Logic. Being an Essay Towards a Calculus of Deductive Reasoning*, Cambridge, reprinted, Oxford, 1951.
Crapo, H. H. and G.-C. Rota [1970a] *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, MIT Press, Cambridge, Mass.
Crapo, H. H. and G.-C. Rota [1970b] Geometric lattices, in: *Trends in Lattice Theory*, (ed. J. C. Abbott), van Nostrand-Reinhold, New York, pp. 127–172.
Crawley, P. and R. P. Dilworth [1973] *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, N.J.
Czédli, G. [1999] *Lattice Theory* (Hungarian), JATEPress, Szeged.
Dauben, J. W. [1986] Review of Mehrtens [1979], Order 3, 89–102.
Davey, B. A. and H. A. Priestley [1990] *Introduction to Lattices and Order*, Cambridge Univ. Press, Cambridge.
Dedekind, R. [1900] Über die von drei Moduln erzeugte Dualgruppe, Math. Ann. 53, 371–403, Gesammelte Werke,Vol. 2, pp. 236–271.
Dilworth, R. P. [1940] Lattices with unique irreducible decompositions, Ann. of Math. 41, 771–777.
Dilworth, R. P. [1941a] The arithmetical theory of Birkhoff lattices, Duke Math. J. 8, 286–299.
Dubreil-Jacotin, M. L., L. Lesieur, and R. Croisot [1953] *Leçons sur la théorie des treillis, des structures algébriques ordonnées et des treillis géométriques*, Gauthier-Villards, Paris.
Grätzer, G. [1971] *Lattice Theory: First Concepts and Distributive Lattices*, Freeman, San Francisco.
Grätzer, G. [1978] *General Lattice Theory*, Birkhäuser, Basel.
Grätzer, G. [1979] *Universal Algebra* (2nd edition), Springer-Verlag, New York.
Grätzer, G. [1998] *General Lattice Theory*, Birkhäuser, Basel. Second, completely revised edition.
Halmos, P. R. [1963] *Lectures on Boolean Algebras*, Van Nostrand, Princeton, N.J.
Hibi [1992] *Algebraic Combinatorics on Convex Polytopes*, Carslaw, Glebe, Australia.
Kung, J. P. S. [1986a] *A Source Book in Matroid Theory*, Birkhäuser, Boston.
Kurosch, A. G. [1935] Durchschnittsdarstellungen mit irreduziblen Komponenten in Ringen und in sogenannten Dualgruppen, Mat. Sbornik 42, 613–616.
Mac Lane, S. [1938] A lattice formulation for transcendence degrees and p-bases, Duke Math. J. 4, 455–468.
Mac Lane, S. [1976] Topology and logic as a source of algebra, Bull. Amer. Math. Soc. 82, 1–40.
Maeda, F. [1958] *Kontinuierliche Geometrien*, Springer-Verlag, Berlin.
Maeda, F. and S. Maeda [1970] *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.
Mangione, C. and S. Bozzi [1993] *Storia della Logica da Boole ai Nostri Giorni*, Garzanti Editore, Milano.
McKenzie, R., G. F. McNulty, and W. Taylor [1987] *Algebras, Lattices, Varieties*, Vol. 1, Wadsworth & Brooks/Cole, Monterey, Calif.
Mehrtens, H. [1979] *Die Entstehung der Verbandstheorie*, Gerstenberg, Hildesheim.
Menger, K. [1936] New foundations of projective and affine geometry, Ann. of Math. 37, 456–482.
Noether, E. [1921] Idealtheorie in Ringbereichen, Math. Ann. 83, 24–66.
Ore, O. [1935] On the foundations of abstract algebra I, Ann. of Math. 36, 406–437.
Ore, O. [1936] On the foundations of abstract algebra II, Ann. of Math. 37, 265–292.
Rota, G.-C. and J. S. Oliveira (eds.) [1987] *Selected Papers on Algebra and Topology by Garrett Birkhoff*, Birkhäuser, Boston.
Sikorski, R. [1964] *Boolean Algebras* (2nd edition), Academic Press, New York.
Skornjakov, L. A. [1961] *Complemented Modular Lattices and Regular Rings* (Russian), Gosudarst. Izdat. Fiz.-Mat. Lit., Moscow.

Stanley, R. [1986] *Enumerative Combinatorics I*, Wadsworth & Brooks/Cole, Monterey, Calif.
Szász, G. [1963] *Introduction to Lattice Theory*, Akadémiai Kiadó, Budapest, and Academic Press, New York.
Topping, D. M. [1967] Asymptoticity and semimodularity in projection lattices, Pacific J. Math. 20, 317–325.
von Neumann, J. [1960] *Continuous Geometry* (ed. I. Halperin), Princeton Mathematical Ser. 25, Princeton Univ. Press, Princeton, N.J.
White, N. L. (ed.) [1986] *Theory of Matroids*, Encyclopedia of Mathematics and Its Applications, Vol. 26, Cambridge Univ. Press, Cambridge.
White, N. L. (ed.) [1987] *Combinatorial Geometries*, Encyclopedia of Mathematics and Its Applications, Vol. 29, Cambridge Univ. Press, Cambridge.
White, N. L. (ed.) [1992] *Matroid Applications*, Encyclopedia of Mathematics and Its Applications, Vol. 40, Cambridge Univ. Press, Cambridge.
Whitney, H. [1935] On the abstract properties of linear dependence, Amer. J. Math. 57, 509–533.
Wilcox, L. R. [1938] Modularity in the theory of lattices, Bull. Amer. Math. Soc. 44, 50.
Wilcox, L. R. [1939] Modularity in the theory of lattices, Ann. of Math. 40, 490–505.

1.2 Boolean Lattices, Ortholattices, and Orthomodular Lattices

Summary. We recall the definitions of Boolean lattices, distributive lattices, modular lattices, and orthocomplemented and orthomodular lattices. The notion of modular pair is introduced. Some important results and examples are given.

A lattice is called *distributive* if

(D) $c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b)$

holds for all triples (a, b, c) of lattice elements. A lattice is distributive if and only if $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$ holds for all triples (a, b, c) of lattice elements.

For the visual representation of posets and lattices we frequently use Hasse diagrams. The lattices in Figure 1.2(a), (d) are distributive, whereas the lattices in Figure 1.2(b), (c) are not. The lattices in Figure 1.2(b) and (c) will be denoted by M_3 and N_5, respectively.

Let us explain some more concepts. We say that x is a *lower cover* of y and we write $x \prec y$ if $x < y$ and $x \leq t < y$ implies $t = x$. Equivalently we say in this case that y is an *upper cover* of x and write $y \succ x$. If a lattice has a *least element*, denoted by 0, we also say that the lattice is *bounded below*. If a lattice has a *greatest element*, denoted by 1, we also say that the lattice is *bounded above*. A *bounded lattice* is a lattice having both a least element and a greatest element. In a lattice bounded below, an upper cover of the least element is called an *atom*. In Figure 1.2(c) the elements a and c are atoms, but b is not. A lattice bounded below is said to be *atomistic* if every of its elements ($\neq 0$) is a join of atoms. The lattice in Figure 1.2(b) is atomistic, but the lattices in Figure 1.2(a), (c), and (d) are not. In a lattice bounded below an element z ($\neq 0$) is called a *cycle* if the interval $[0, z]$ is a chain. Every atom is a cycle; in Figure 1.2(c) the element b is a

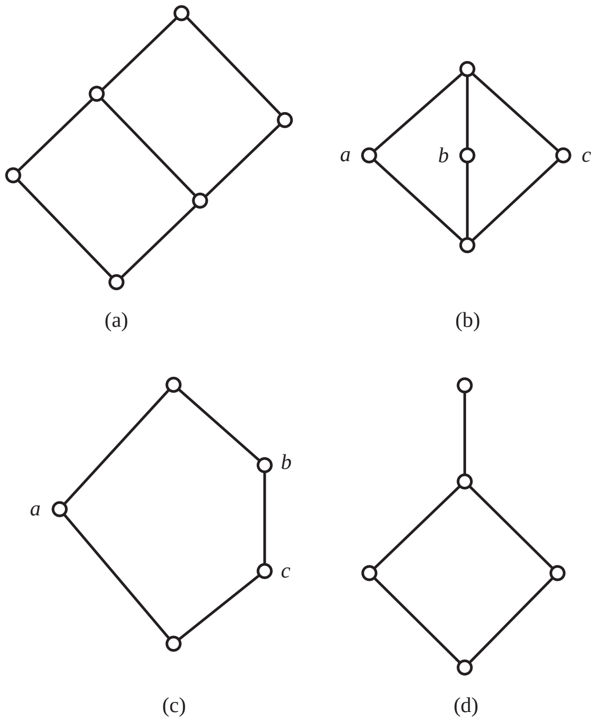


Figure 1.2

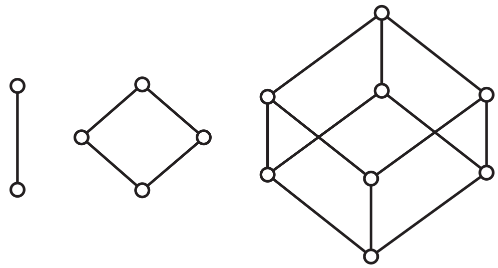
cycle that is not an atom. A lattice bounded below is called *cyclically generated* if every element ($\neq 0$) is a join of cycles. The lattices in Figure 1.2(a), (b), (c) are cyclically generated, but the lattice in Figure 1.2(d) is not. Any atomistic lattice is cyclically generated. The name *cycle* has its origin in the theory of abelian groups. Figure 1.2(a) is isomorphic to the lattice of all subgroups of the cyclic group Z_{18}. Figure 1.2(b) is isomorphic to the the subgroup lattice of the Klein 4-group.

A sublattice K of a lattice L is called a *diamond* or a *pentagon* if K is isomorphic to M_3 or N_5, respectively. Distributivity is characterized by the absence of diamonds and pentagons:

Theorem 1.2.1 *A lattice is distributive if and only if it does not contain a diamond or a pentagon.*

This is Birkhoff's distributivity criterion (Birkhoff [1934]). For more on distributive lattices see Section 1.3.

In a bounded lattice, an element $\overline{a}$ is a complement of a if $a \wedge \overline{a} = 0$ and $a \vee \overline{a} = 1$. A *complemented lattice* is a bounded lattice in which every element has a complement. The lattices of Figure 1.2(b) and (c) are complemented, whereas



(a) $\mathbf{2}^1$ (b) $\mathbf{2}^2$ (c) $\mathbf{2}^3$

Figure 1.3

the lattices in Figure 1.2(a) and (d) are not. A complemented distributive lattice will be called a *Boolean lattice*. In a Boolean lattice B, every element has a unique complement and B is also *relatively complemented*, that is, every interval of B is a complemented sublattice.

We shall distinguish Boolean lattices from Boolean algebras. A *Boolean algebra* is a Boolean lattice in which the least element 0, the greatest element 1 and the complementation $\overline{}$ are also considered to be operations. In other words, a Boolean algebra is a system $\mathbf{B} = \langle B, \wedge, \vee, \overline{}, 0, 1\rangle$ with the two binary operations $\wedge, \vee$, the unary operation $\overline{}$, and the nullary operations 0 and 1. We use the standard notation $\mathbf{2}^n (n = 1, 2, 3, \ldots)$ for the Boolean lattice consisting of 2^n elements. The Boolean lattices $\mathbf{2}^1$, $\mathbf{2}^2$, and $\mathbf{2}^3$ are shown in Figure 1.3(a), (b), and (c), respectively.

Any distributive lattice obviously satisfies the implication

(M) $c \leq b \Rightarrow c \vee (a \wedge b) = (c \vee a) \wedge b$

for all elements a, b, c. A lattice is called *modular* if it satisfies (M) for all a, b, c. Modularity is the most important generalization of distributivity. The diamond M_3 [Fig. 1.2(b)] is modular but not distributive. The pentagon N_5 [Fig. 1.2(c)] is not modular. Modularity can be characterized by the absence of pentagons:

Theorem 1.2.2 *A lattice is modular if and only if it does not contain a pentagon.*

This characterization is due to Dedekind [1900]. Some more properties of modular lattices will be given in Section 1.6. However, let us note here that for some special classes of lattices, the forbidden-sublattice characterizations for modular and distributive lattices can be sharpened by showing the existence of very large or very small pentagons or diamonds.

For instance, a bounded relatively complemented nonmodular lattice always contains a pentagon as $\{0, 1\}$-sublattice. The same is true of the diamond in certain complemented modular lattices (von Neumann [1936–7]). If a lattice is finite and nonmodular, then the pentagon it contains can be required to satisfy $b \succ c$ [the notation referring to Fig. 1.2(c)]. The modularity criterion (Theorem 1.2.2) simplifies as follows in the case of complemented atomic lattices (a lattice L with 0 is called *atomic* if for every $x \in L$, $x \neq 0$, there exists an atom $p \in L$ such that $x \geq p$).

Theorem 1.2.3 *If a complemented atomic lattice contains no pentagon including both the least and greatest elements, then the lattice is modular.*

This theorem is due to McLaughlin [1956]. For a proof we also refer to Salii [1988], pp. 27–30, and to Dilworth [1982], pp. 333–353.

In nonmodular lattices we shall be interested in modular pairs and dual modular pairs, which were introduced by Wilcox [1938], [1939]. We say that an ordered pair (a, b) of elements of a lattice L is a *modular pair* and we write $a\ M\ b$ if, for all $c \in L$,

$$c \leq b \quad \text{implies} \quad c \vee (a \wedge b) = (c \vee a) \wedge b.$$

We say that (a, b) is a *dual modular pair* and we write $a\ M^*\ b$ if, for all $c \in L$,

$$c \geq b \quad \text{implies} \quad c \wedge (a \vee b) = (c \wedge a) \vee b.$$

If (a, b) is not a modular pair, then we write $a\ \overline{M}\ b$. It is clear that a lattice is modular if and only if every ordered pair of elements is modular. In the nonmodular lattice of Figure 1.2(c) we have $b\ M\ a$ but $a\ \overline{M}\ b$, which shows that the relation of being a modular pair is not symmetric in this lattice. Similarly, this example also shows that the relation of being a dual modular pair is not symmetric in general.

Let L be a lattice with 0 and 1. An *orthocomplementation* on L is a unary operation $a \to a^\perp$ on L satisfying the following three conditions:

(i) $a \wedge a^\perp = 0$, $a \vee a^\perp = 1$, that is, $a^\perp$ is a complement of a;
(ii) $a \leq b$ implies $b^\perp \leq a^\perp$;
(iii) $a^{\perp\perp} = a$ for every $a \in L$.

Note that $a^{\perp\perp}$ stands for $(a^\perp)^\perp$. We call $a^\perp$ the *orthocomplement* of a. An *orthocomplemented lattice* (briefly: *ortholattice*, *OC lattice*) is a lattice with 0 and 1 carrying an orthocomplementation.

Any Boolean lattice is an ortholattice (the Boolean complement of an element being its orthocomplement). Two other examples of ortholattices are shown in Figure 1.4. The “benzene ring” of Figure 1.4(a) is also be called the *hexagon*. The lattice in Figure 1.4(b) is the *horizontal sum* of the *blocks* $\mathbf{2}^2$ and $\mathbf{2}^3$, that is, $\mathbf{2}^2 \cap \mathbf{2}^3 = \{0, 1\}$.

For elements a, b of an ortholattice the *De Morgan laws*

$$(a \vee b)^\perp = a^\perp \wedge b^\perp \quad \text{and} \quad (a \wedge b)^\perp = a^\perp \vee b^\perp$$

hold, since the orthocomplementation $a \to a^\perp$ is a dual isomorphism of the lattice onto itself. Conversely, either of the two De Morgan laws implies (i) in the

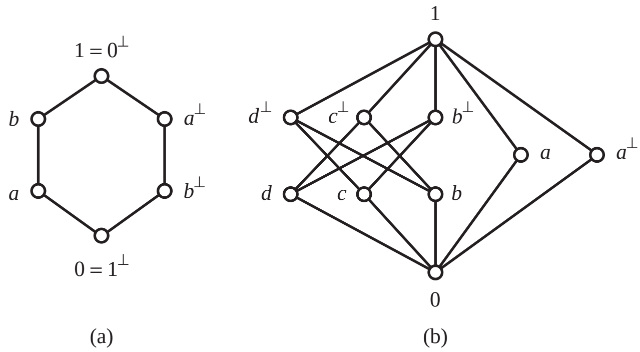


Figure 1.4

preceding definition of ortholattices. Hence we may replace (i) by $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$ or by its dual. For elements a, b of an ortholattice we define a binary relation $\perp$ ("orthogonality") by

$$a \perp b \quad \text{if and only if} \quad a \leq b^{\perp}.$$

If $a \perp b$ holds, we also say that a is *orthogonal* to b.

Using the above-introduced orthogonality relation and the notion of modular pairs, we now define orthomodular lattices. An orthocomplemented lattice L is called *orthomodular* if, for all $a, b \in L$, $a \perp b$ implies $a\ M\ b$. In other words, every orthogonal pair is a modular pair. This explains the expression *orthomodular*.

Orthomodularity can be characterized in several ways within the class of orthocomplemented lattices. Some of these characterizations are gathered in

Theorem 1.2.4 *In an orthocomplemented lattice L the following five statements are equivalent:*

(i) *L is orthomodular;*
(ii) *$a\ M\ a^{\perp}$ holds for all $a \in L$;*
(iii) *$a\ M^{*}\ a^{\perp}$ holds for all $a \in L$;*
(iv) *$a \leq b$ implies $a \vee (a^{\perp} \wedge b) = b$;*
(v) *$a \leq b$ implies the existence of $c \in L$ such that $a \perp c$ and $a \vee c = b$.*

For a proof see Maeda & Maeda [1970], Theorem 29.13, p. 132. Any Boolean lattice is orthomodular. The lattice in Figure 1.4(a) is the simplest example of an orthocomplemented lattice that is not orthomodular. The lattice of all subspaces of the three-dimensional real Euclidean space is modular and orthocomplemented and hence orthomodular. The lattice in Figure 1.4(b) is orthomodular, but not modular.

The lattices in Figures 1.2–1.4 are finite and hence of finite length. Let us now give examples of orthomodular lattices of infinite length. (We refer to Section 1.9 for a formal definition of the notion of length.)

Our first example is the lattice of closed subspaces of a Hilbert space. Let H denote a Hilbert space. As a metric space, H is complete, which means that every Cauchy sequence in H converges. A subspace M of H is *closed* if for every Cauchy sequence γ_m in M with $\gamma_m \to x \in H$ implies $x \in M$. A closed subspace is also called a *flat*. By $L_c(H)$ we denote the set of all closed subspaces of H. The lattice $\langle L_c(H); \subseteq \rangle$ is a complete lattice $\langle L_c(H); \wedge, \vee \rangle$, the meet of two closed subspaces being their set-theoretic intersection and the join being the closure of their sum. This lattice will also briefly be denoted by $L_c(H)$. For a subspace M of H we define $M^\perp = \{x \in H : \langle x, y \rangle = 0 \text{ for all } y \in M\}$ with $\langle x, y \rangle$ denoting the inner product of x and y defined on H. The operation $M \to M^\perp$ is an orthocomplementation, and $L_c(H)$ is a complete orthomodular lattice (Sasaki [1954]). We can interpret the orthomodular identity as a basic fact about the geometry of Hilbert space: If the closed subspace M is contained in the closed subspace N, then N is the *orthogonal direct sum* of M and $N - M$ (the orthocomplement of M in N), that is, $M \le N$ implies $N = M \oplus (N - M)$. For more details see for example Halmos [1957]. The lattice of all closed subspaces of an infinite-dimensional Hilbert space H is atomistic but not modular. Indeed, one can show that the lattice $L_c(H)$ is modular if and only if H is finite-dimensional.

Without going into details, we mention as a second example the projection lattice of a von Neumann algebra. This lattice is orthomodular, but neither modular nor atomistic. For some more information see Section 2.6. There are several monographs dealing with the theory and applications of orthomodular lattices, such as Maeda & Maeda [1970], Kalmbach [1983], [1986], Beran [1984]. Other references will be given in the Notes below and in Section 2.6.

We close this section with a brief look at varieties of lattices.[1] Let $p_i = q_i$ be identities for $i \in I$. The class K of all lattices satisfying all identities $p_i = q_i$, $i \in I$, is called a *variety* (or *equational class*) of lattices. A variety is *trivial* if and only if it contains one-element lattices only. Let K be a class of lattices. We use the following notation:

$\boldsymbol{H}(\mathsf{K})$ denotes the class of all homomorphic images of members of K.
$\boldsymbol{S}(\mathsf{K})$ denotes the class of all sublattices of members of K.
$\boldsymbol{P}(\mathsf{K})$ denotes the class of all isomorphic images of direct products of members of K.

We say that K is *closed under the formation of homomorphic images, under the formation of sublattices,* and *under the formation of direct products* if $\boldsymbol{H}(\mathsf{K}) \subseteq \mathsf{K}$, $\boldsymbol{S}(\mathsf{K}) \subseteq \mathsf{K}$, and $\boldsymbol{P}(\mathsf{K}) \subseteq \mathsf{K}$, respectively. The following result is known as Birkhoff's ***HSP*** theorem (Birkhoff [1935b]).

[1] For this topic see the monograph *Varieties of Lattices* by P. Jipsen and H. Rose, Springer-Verlag, Berlin (1992).

Theorem 1.2.5 *A class* K *of lattices is a variety if and only if* K *is closed under the formation of homomorphic images, under the formation of sublattices, and under the formation of direct products.*

Corollary 1.2.6 *Let* K *be a class of lattices. Then* ***HSP*** (K) *is the smallest variety containing* K.

The corollary is due to Tarski [1946]. The *smallest variety containing* K will be denoted by $\boldsymbol{V}(\mathsf{K})$. We shall also say that $\boldsymbol{V}(\mathsf{K})$ is *the variety generated by* K. Several important classes of lattices form a variety. We have already mentioned the class of one-element lattices (the trivial variety); the class L of all lattices is also a variety. It is immediate from the definition that distributive lattices form a variety. A Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \bar{\ }, 0, 1\rangle$ can be defined by equations. Hence the class of Boolean algebras is a variety. An orthomodular lattice considered as an algebra $\mathbf{L} = \langle L, \wedge, \vee, ^{\perp}, 0, 1\rangle$ can be defined by equations. Hence the class of orthomodular lattices is a variety.

There are many statements equivalent to modularity. It can be shown, for example, that a lattice $\mathbf{L} = \langle L, \wedge, \vee\rangle$ is modular if and only if it satisfies $((b \wedge c) \vee a) \wedge c = (b \wedge c) \vee (a \wedge c)$ for all $a, b, c \in L$. Hence the class of modular lattices is a variety. In contrast to this, the class of semimodular lattices does not form a variety (see Section 1.7).

Notes

The theory of orthomodular lattices has its roots in functional analysis, and its origins go back to the theory of von Neumann algebras. For a thorough study of this background and an analysis of the historical sequence von Neumann algebras $\to$ continuous geometries $\to$ orthomodular lattices we refer to Holland [1970]. The theory of continuous geometries was also invented by von Neumann and developed in the period 1935–7 (for von Neumann's contribution to lattice theory see Birkhoff [1958]).

The investigation of the lattice-theoretical foundations of quantum-mechanical systems was initiated by Birkhoff & von Neumann [1936], who set up a model of what they called the *logic of quantum mechanics*. However, their lattices were modular, and this turned out to be too restrictive a condition. The interpretation of observables as operators in Hilbert space and, in particular, the investigation of lattices of projection operators led to nonmodular orthomodular lattices.

The relationship between certain orthomodular lattices (or more general structures) and quantum mechanics belongs to the vast field vaguely described as *quantum logics*. A popular account of some problems arising in this field is McGrath [1991]. For details we refer to the monographs Beltrametti & Cassinelli [1981], Cohen [1989], and Pták & Pulmannová [1991].

References

Beltrametti, E. G. and G. Cassinelli [1981] *The Logic of Quantum Mechanics*, Encyclopedia of Mathematics and its Applications, Vol. 15, Addison-Wesley, Reading, Mass.

Beran, L. [1984] *Orthomodular Lattices. Algebraic Approach*, Reidel, Dordrecht.

Birkhoff, G. [1934] Applications of lattice algebra, Proc. Camb. Phil. Soc. 30, 115–122.

Birkhoff, G. [1935b] On the structure of abstract algebras, Proc. Camb. Phil. Soc. 31, 433–454.

Birkhoff, G. [1958] von Neumann and lattice theory, in: *John von Neumann, 1903–1957*, Bull. Amer. Math. Soc. 64, 50–56.

Birkhoff, G. and J. von Neumann [1936] The logic of quantum mechanics, Ann. of Math. 37, 823–843.

Cohen, D.W. [1989] *An Introduction to Hilbert Space and Quantum Logic*, Springer-Verlag, New York.

Dedekind, R. [1900] Über die von drei Moduln erzeugte Dualgruppe, Math. Ann. 53, 371–403; *Gesammelte Werke*, Vol. 2, pp. 236–271.

Dilworth, R. P. [1982] The role of order in lattice theory, in: Rival [1982a], pp. 333–353.

Grätzer, G. [1978] *General Lattice Theory*, Birkhäuser, Basel.

Halmos, P. R. [1957] *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea, New York.

Holland, S. S., Jr. [1970] The current interest in orthomodular lattices, in: *Trends in Lattice Theory* (ed. J.C. Abbott), van Nostrand Reinhold, New York, pp. 41–126.

Kalmbach, G. [1983] *Orthomodular Lattices*, Academic Press, London.

Kalmbach, G. [1986] *Measures and Hilbert Lattices*, World Scientific, Singapore.

Maeda, F. and S. Maeda [1970] *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.

McGrath, J. H. [1991] Quantum theory and the lattice join, Math. Intelligencer 13, 72–79.

McLaughlin, J. E. [1956] Atomic lattices with unique comparable complements, Proc. Amer. Math. Soc. 7, 864–866.

Pták, P. and S. Pulmannová [1991] *Orthomodular Structures as Quantum Logics*, Kluwer Academic, Dordrecht.

Salii, V. N. [1988] *Lattices with Unique Complements*, Translations of Mathematical Monographs, Vol. 69, Amer. Math. Soc., Providence, R.I. (translation of the 1984 Russian edition, Nauka, Moscow).

Sasaki, U. [1954] Orthocomplemented lattices satisfying the exchange axiom, J. Sci. Hiroshima Univ. A 17, 293–302.

Tarski, A. [1946], A remark on functionally free algebras, Ann. of Math. 47, 163–165.

von Neumann, J. [1936–7] *Lectures on Continuous Geometries*, Inst. for Advanced Study, Princeton, N.J.

Wilcox, L. R. [1938] Modularity in the theory of lattices, Bull. Amer. Math. Soc. 44, 50.

Wilcox, L. R. [1939] Modularity in the theory of lattices, Ann. of Math. 40, 490–505.

1.3 Distributive and Semidistributive Lattices

Summary. The early work on distributive lattices included representation theorems, embedding theorems, and structure theorems. Later it was discovered that the congruence lattice of a lattice and the lattice of lattice varieties are distributive. We mention some of these results with a view to later applications and generalizations. We also give some facts concerning semidistributivity, which is next to modularity the most important generalization of distributivity.

An element j (m) of a lattice L is called *join-irreducible* (*meet-irreducible*) if, for all $x, y \in L$,

$$j = x \vee y \text{ implies } j = x \text{ or } j = y \qquad (m = x \wedge y \text{ implies } m = x \text{ or } m = y).$$

For a lattice L of finite length let $J(L)$ denote the set of all nonzero join-irreducible elements, regarded as a poset under the partial ordering of L. Sometimes we emphasize this by writing more precisely $(J(L), \leq)$ instead of $J(L)$. By j' we denote the uniquely determined lower cover of $j \in J(L)$. For a poset P, we call a subset $I \subseteq P$ an *order ideal* (or *hereditary subset*) if $x \in I$ and $y \leq x$ imply $y \in I$. Let ord(P) denote the set of all order ideals of the poset P, and regard ord(P) as partially ordered by set inclusion. With respect to this partial order, ord(P) forms a lattice in which meet and join are the set-theoretic intersection and union, respectively. Thus ord(P) is a distributive lattice. For finite distributive lattices we have the following structural result

Theorem 1.3.1 *Let L be a finite distributive lattice. Then the map*

$$\varphi : a \to \{j : j \leq a, \ j \in J(L)\} = (a] \cap J(L)$$

is an isomorphism between L and ord($J(L)$).

For a proof see Grätzer [1978], pp. 61–62, or Stanley [1986], p. 106.

Corollary 1.3.2 *The correspondence $L \to J(L)$ makes the class of all finite distributive lattices (with more than one element) correspond to the class of all finite posets. Isomorphic lattices correspond to isomorphic posets, and vice versa.*

The preceding corollary is called the *fundamental theorem on finite distributive lattices*. It has several important consequences. Recall that a subset S of the power set of a set is called a *ring of sets* if $X, Y \in S$ implies both $X \cap Y \in S$ and $X \cup Y \in S$. Since ord($J(L)$) is a ring of sets, we have

Corollary 1.3.3 *A finite lattice is distributive if and only if it is isomorphic to a ring of sets.*

In particular, we have

Corollary 1.3.4 *A finite lattice is Boolean if and only if it is isomorphic to the lattice of all subsets of a finite set.*

If a is an element of a lattice L, then a representation $a = j_1 \vee \cdots \vee j_n$ of a as a join of finitely many join-irreducible elements $j_1, \ldots, j_n \in J(L)$ is called a *finite join decomposition* of a. This join decomposition is said to be *irredundant* if, for each $i = 1, \ldots, n$, one has $a \neq j_1 \vee \cdots \vee j_{i-1} \vee j_{i+1} \vee \cdots \vee j_n$. Dually one defines *finite meet decompositions* and *irredundant finite meet decompositions* (see Crawley & Dilworth [1973], p. 38). Finite join decompositions or finite meet decompositions do not always exist. However, if a lattice element has a finite join decomposition, then it clearly has an irredundant join decomposition, which is obtained by omitting superfluous elements from the given join decomposition. A similar statement holds for finite meet decompositions. It is easy to see that in a distributive lattice an element has at most one irredundant join decomposition and at most one irredundant meet decomposition. In the finite case we have

Corollary 1.3.5 *Every element of a finite distributive lattice has a unique irredundant join decomposition and a unique irredundant meet decomposition.*

Similar results have also been proved for distributive lattices that are not finite but satisfy certain relaxations of finiteness. For example, if a distributive lattice satisfies the *ascending chain condition* (ACC for short), then each of its elements has a unique irredundant meet decomposition (see Birkhoff [1948], p. 142), which is necessarily finite. In Section 1.8 we shall have a look at the existence of possibly infinite meet decompositions and the question of irredundant meet decompositions. The uniqueness property for irredundant meet decompositions will be discussed in more detail in Chapter 7.

Let $M(L)$ denote *the set of meet irreducible elements* ($\neq 1$) of a lattice of finite length L. By m^* we denote the uniquely determined upper cover of $m \in M(L)$. In a finite distributive lattice there is a natural one-to-one correspondence between the meet irreducibles ($\neq 1$) and the join irreducibles ($\neq 0$): For every $m \in M(L)$, there exists a unique minimal join irreducible $j \in J(L)$ such that $j \not\leq m$. In turn, m is the unique maximal meet irreducible not containing j. This correspondence implies

Corollary 1.3.6 *If L is a finite distributive lattice, then $|J(L)| = |M(L)|$.*

For a proof see Grätzer [1978], pp. 62–63. For a finite distributive lattice L it is even true that $J(L) \cong M(L)$. More precisely, we have $(J(L), \leq) \cong (M(L), \leq)$, where $\leq$ denotes the partial ordering induced by L (see Pezzoli [1984] for a proof). This property of finite distributive lattices is illustrated in Figure 1.5.

Consider now Con(L), the set of congruence relations on a lattice L, and let Σ be a subset of Con(L). Define the relation π in L by $a\ \pi\ b$ if $a\ \theta\ b$ holds for all $\theta \in \Sigma$, and define the relation σ by the rule $a\,\sigma\,b$ if there exist a sequence $a = a_0, a_1, \ldots, a_n = b$ in L and congruence relations $\theta_1, \ldots, \theta_n \in \Sigma$ such that $a_{i-1}\theta_i a_i$ for each $i = 1, \ldots, n$. It is easy to see that π and σ are congruence relations, that π is the meet in Con(L) of the subset Σ, and that σ is the join in Con(L) of Σ. Hence Con(L) is a complete lattice. Funayama & Nakayama [1942] proved that the lattice Con(L) of congruence relations on a lattice L is distributive and algebraic. For a proof see also Crawley & Dilworth [1973], p. 75. The fact that, for an arbitrary lattice L, the congruence lattice Con(L) is distributive can be reformulated by saying that the variety of all lattices is *congruence-distributive*.

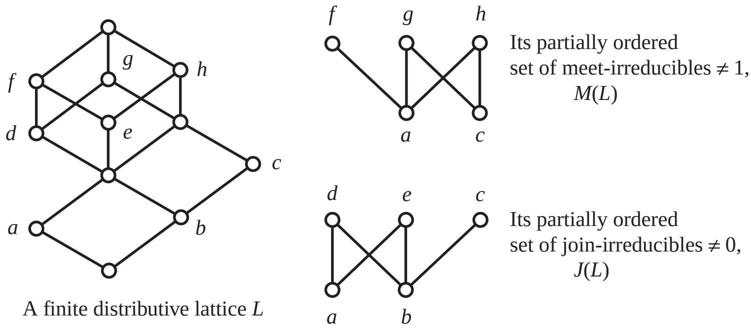


Figure 1.5

We have already remarked that modularity is the most important generalization of distributivity. There are weakenings of distributivity going in other directions than modularity. Among these, semidistributivity turned out to be particularly fruitful. A lattice L is called *meet-semidistributive* if

(SD∧) $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$ for all $a, b, c \in L$.

A lattice L is called *join-semidistributive* if

(SD∨) $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$ for all $a, b, c \in L$,

and a lattice is called *semidistributive* – (SD) for short – if it satisfies both (SD∧) and (SD∨). In what follows we briefly recall some results on (SD), (SD∧), and (SD∨) with an eye to later applications (e.g. in Section 9.3).

Semidistributivity was introduced by Jónsson [1961] in his investigations of free lattices.[2] Jónsson proved that (SD∧) and (SD∨) hold in a free lattice. Hence any sublattice of a free lattice is semidistributive. Let us note that sublattices of free lattices also satisfy the following condition due to Whitman [1941]:

(W) $x \wedge y \leq u \vee v$ implies $[x \wedge y, u \vee v] \cap \{x, y, u, v\} \neq \emptyset$.

Nation [1982] proved Jónsson's longstanding conjecture that a finite lattice is isomorphic to a sublattice of a free lattice if and only if it is a semidistributive lattice satisfying *Whitman's condition* (W). Free lattices provide the most important examples of semidistributive lattices. However, semidistributivity has also shown up in other areas of lattice theory. In particular, meet semidistributivity appears in the congruence lattice of meet semilattices, and it plays an important role in the study of lattice varieties. Let us give some more examples. We begin with lattices that are *not* semidistributive.

The lattice M_3 is not semidistributive; in fact, it satisfies neither (SD∧) nor (SD∨). Next we consider the lattices S_7, S_7^*, L_3, L_4, L_5 shown in Figure 1.6. From

[2]For this topic see the monograph *Free Lattices* by R. Freese, J. Ježek, and J. B. Nation Amer. Math. Soc., Providence, R.I. (1995).