Combinatorial Games

Traditional game theory has been successful at developing strategy in games of incomplete information: when one player knows something that the other does not. But it has little to say about games of complete information, for example Tic-Tac-Toe, solitaire, and hex. This is the subject of Combinatorial Game Theory. Most board games are a challenge for mathematics: to analyze a position one has to examine the available options, and then the further options available after selecting any option, and so on. This leads to combinatorial chaos, where brute force study is impractical.

In this comprehensive volume, József Beck shows readers how to escape from the combinatorial chaos via the fake probabilistic method, a game-theoretic adaptation of the probabilistic method in combinatorics. Using this, the author is able to determine the exact results about infinite classes of many games, leading to the discovery of some striking new duality principles.

József Beck is a Professor in the Mathematics Department of Rutgers University. He has received the Fulkerson Prize for research in Discrete Mathematics and has written around 100 research publications. He is the co-author, with W. L. Chen, of the pioneering monograph *Irregularities of Distribution.*

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Combinatorial Games

Tic-Tac-Toe Theory

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521461009

First published 2008

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

ISBN 978-0-521-46100-9 hardback

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Frontmatter
More information

Dedicated to
my mother who taught me how to play Nine Men's Morris ("Mill")

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
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Preface

There is an old story about the inventor of Chess, which goes something like this. When the King learned the new game, he quickly fell in love with it, and invited the inventor to his palace. "I love your game," said the King, "and to express my appreciation, I decided to grant your wish." "Oh, thank you, Your Majesty," began the inventor, "I am a humble man with a modest wish: just put one piece of rice on the first little square of the chess board, 2 pieces of rice on the second square, 4 pieces on the third square, 8 pieces on the fourth square, and so on; you double in each step." "Oh, sure," said the King, and immediately called for his servants, who started to bring in rice from the huge storage room of the palace. It didn't take too long, however, to realize that the rice in the palace was not enough; in fact, as the court mathematician pointed out, even the rice produced by the whole world in the last thousand years wouldn't be enough to fulfill the inventor's wish ($2^{64} - 1$ pieces of rice). Then the King became so angry that he gave the order to execute the inventor. This is how the King discovered Combinatorial Chaos.

Of course, there is a less violent way to discover Combinatorial Chaos. Any attempt to analyze unsolved games like Chess, Go, Checkers, grown-up versions of Tic-Tac-Toe, Hex, etc., lead to the same conclusion: we get quickly lost in millions and millions of cases, and feel shipwrecked in the middle of the ocean.

To be fair, the hopelessness of Combinatorial Chaos has a positive side: it keeps the games alive for competition.

Is it really hopeless to escape from Combinatorial Chaos? The reader is surely wondering: "How about Game Theory?" "Can Game Theory help here?" Traditional Game Theory focuses on games of *incomplete information* (like Poker where neither player can see the opponent's cards) and says very little about Combinatorial Games such as Chess, Go, etc. Here the term Combinatorial Game means a 2-player zero-sum game of skill (no chance moves) with *complete information*, and the payoff function has 3 values only: win, draw, and loss.

The "very little" that Traditional Game Theory can say is the following piece of advice: try a backtracking algorithm on the game-tree. Unfortunately, backtracking leads to mindless exponential-time computations and doesn't give any insight; this is better than nothing, but not much. Consequently, computers provide remarkably little help here; for example, we can easily simulate a *random play* on a computer, but it is impossible to simulate an *optimal play* (due to the enormous complexity of the computations). We simply have no data available for these games; no data to extrapolate, no data to search for patterns.

The 3-dimensional $5 \times 5 \times 5$ version of Tic-Tac-Toe, for instance, has about 3^{125} positions (each one of the 5^3 cells has 3 options: either marked by the first player, or marked by the second player, or unmarked), and backtracking on a graph of 3^{125} vertices ("position graph") takes at least 3^{125} steps, which is roughly the third power of the "chaos" the chess-loving King was facing above. No wonder the $5 \times 5 \times 5 = 5^3$ Tic-Tac-Toe is unsolved!

It is even more shocking that we know only two(!) explicit winning strategies in the whole class of $n \times n \times \cdots \times n = n^d$ Tic-Tac-Toe games: the 3^3 version (which has an easy winning strategy) and the 4^3 version (which has an extremely complicated winning strategy).

If traditional Game Theory doesn't help, and the computer doesn't really help either, then what can we do? The objective of this book is exactly to show an escape from Combinatorial Chaos, to win a battle in a hopeless war. This "victory" on the class of Tic-Tac-Toe-like games is demonstrated. Tic-Tac-Toe itself is for children (a very simple game really), but there are many grown-up versions, such as the $4 \times 4 \times 4 = 4^3$ game, and, in general, the $n \times n \times \cdots \times n = n^d$ hypercube versions, which are anything but simple. Besides hypercube Tic-Tac-Toe, we study Clique Games, Arithmetic Progression Games, and many more games motivated by Ramsey Theory. These "Tic-Tac-Toe-like games" form a very interesting sub-class of Combinatorial Games: these are games for which the standard algebraic methods fail to work. The main result of the book is that for some infinite families of natural "Tic-Tac-Toe-like games with (at least) 2-dimensional goals" we know the exact value of the phase transition between "Weak Win" and "Strong Draw." We call these thresholds Clique Achievement Numbers, Lattice Achievement Numbers, and in the Reverse Games, Clique Avoidance Numbers and Lattice Avoidance Numbers. These are game-theoretic analogues of the Ramsey Numbers and Van der Waerden Numbers. Unlike the Ramsey Theory thresholds, which are hopeless in the sense that the best-known upper and lower bounds are very far from each other, here we can find the exact values of the game numbers. For precise statements see Sections 6, 8, 9, and 12.

To prove these exact results we develop a "fake probabilistic method" (we don't do case studies!); the name *Tic-Tac-Toe theory* in the title of the book actually refers to this "fake probabilistic method." The "fake probabilistic method" has two steps: (1) randomization and (2) derandomization. *Randomization* is a game-theoretic adaptation of the so-called Probabilistic Method ("Erdős Theory"); *derandomization* means to apply potential functions ("resource count"). The Probabilistic Method (usually) gives existence only; the potential technique, on the other hand, supplies explicit strategies. What is more, many of our explicit winning and drawing strategies are very efficient combinatorial algorithms (in fact, the most efficient ones that we know).

The "fake probabilistic method" is not the first theory of Combinatorial Games. There is already a well-known and successful theory: the *addition theory* of "Nim-like compound games." It is an algebraic theory designed to handle complicated games which are, or eventually turn out to be, compounds of several very simple games. "Nim-like compound games" is the subject of the first volume of the remarkable *Winning Ways for your Mathematical Plays* written by *Berlekamp, Conway*, and *Guy* (published in 1982). Volume 1 was called *Theory*, and volume 2 had the more prosaic name of *Case Studies*. As stated by the authors: "there are lots of games for which the theories we have now developed are useful, and even more for which they are not." The family of Tic-Tac-Toe-like games – briefly discussed in Chapter 22 of the *Winning Ways* (vol. 2) – definitely belongs to this latter class. By largely extending Chapter 22, and systematically using the "fake probabilistic method" – which is completely missing(!) from the *Winning Ways* – in this book an attempt is made to upgrade the Case Studies to a Quantitative Theory.

The algebraic and probabilistic approaches represent two entirely different viewpoints, which apparently complement each other. In contrast to the *local* viewpoint of the addition theory, the "fake probabilistic method" is a *global* theory for games which do *not* decompose into simple sub-games, and remain as single coherent entities throughout play. A given position P is evaluated by a score-system which has some natural probabilistic interpretation such as the "loss probability in the randomized game starting from position P." Optimizing the score-system is how we cut short the exhaustive search, and construct efficient ("polynomial time") strategies.

The "fake probabilistic method" works best for large values of the parameters – a consequence of the underlying "laws of large numbers." The "addition theory," on the other hand, works best for little games.

The pioneering papers of the subject are:

1. *Regularity and Positional Games*, by A. W. Hales and R. I. Jewett from 1963;
2. *On a Combinatorial Game*, by P. Erdős and J. Selfridge from 1973;
3. *Biased Positional Games* by V. Chvátal and P. Erdős from 1978; and, as a guiding motivation,
4. the Erdős–Lovász 2-Coloring Theorem from 1975.

The first discovered fundamental connections such as "strategy stealing and Ramsey Theory" and "pairing strategy and Matching Theory", and introduced our basic game class ("positional games"). The last three papers (Erdős with different co-authors) initiated and motivated the "games, randomization, derandomization" viewpoint, the core idea of the book. What is developed here is a far-reaching extension of these ideas – it took 25 years hard labor to work out the details. The majority of the results are published here for the first time.

Being an enthusiastic teacher myself, I tried to write the book in a lecture series format that I would like to use myself in the classroom. Each section is basically an independent lecture; most of them can be covered in the usual 80-minute time frame.

Beside the Theory the book contains dozens of challenging Exercises. The reader is advised to find the solutions to the exercises all by him/herself.

The notation is standard. For example, $c, c_0, c_1, c_2, \ldots$ denote, as usual, positive absolute constants (that I could but do not care to determine); "$a_n = o(1)$" and "$a_n = O(1)$" mean that $a_n \to 0$ and $|a_n| < c$ as $n \to \infty$; and, similarly, "$f(n) = o(g(n))$" and "$f(n) = O(g(n))$" mean that $f(n)/g(n) \to 0$ and $|f(n)/g(n)| < c$ as $n \to \infty$. Also $\log x$, $\log_2 x$, and $\log_3 x$ stand for, respectively, the natural logarithm, the base 2 logarithm, and the base 3 logarithm of x.

There are two informal sections: *A summary of the book in a nutshell* at the beginning, and *An informal introduction to Game Theory* at the end of the book in Appendix D. Both are easy reading; we highly recommend the reader to start the book with these two sections.

Last but not least, I would like to thank the Harold H. Martin Chair at Rutgers University and the National Science Foundation for the research grants supporting my work.