CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

A summary of the book in a nutshell

Mathematics is spectacularly successful at making generalizations: the more than 2000-year old arithmetic and geometry were developed into the monumental fields of calculus, modern algebra, topology, algebraic geometry, and so on. On the other hand, mathematics could say remarkably little about nontraditional complex systems. A good example is the notorious "$3n+1$ problem." If n is even, take $n/2$, if n is odd, take $(3n+1)/2$; show that, starting from an arbitrary positive integer n and applying the two rules repeatedly, eventually we end up with the periodic sequence 1,2,1,2,1,2,.... The problem was raised in the 1930s, and after 70 years of diligent research it is still completely hopeless!

Next consider some games. Tic-Tac-Toe is an easy game, so let's switch to the 3-space. The $3 \times 3 \times 3$ Tic-Tac-Toe is a trivial first player win, the $4 \times 4 \times 4$ Tic-Tac-Toe is a very difficult first player win (computer-assisted proof by O. Patashnik in the late 1970s), and the $5 \times 5 \times 5$ Tic-Tac-Toe is a hopeless open problem (it is conjectured to be a draw game). Note that there is a general recipe to analyze games: perform backtracking on the game-tree (or position graph). For the $5 \times 5 \times 5$ Tic-Tac-Toe this requires about 3^{125} steps, which is totally intractable.

We face the same "combinatorial chaos" with the game of Hex. Hex was invented in the early 1940s by Piet Hein (Denmark), since when it has become very popular, especially among mathematicians. The board is a rhombus of hexagons of size $n \times n$; the two players, White (who starts) and Black, take two pairs of opposite sides of the board. The two players alternately put their pieces on unoccupied hexagons (White has white pieces and Black has black pieces). White (Black) wins if his pieces connect his opposite sides of the board.

In the late 1940s John Nash (*A Beautiful Mind*) proved, by a pioneering application of the Strategy Stealing Argument, that Hex is a first player win. The notorious open problem is to find an *explicit* winning strategy. It remains open for every $n \geq 8$. Note that the standard size of Hex is $n = 11$, which has about 3^{121} different positions.

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

What is common in the $3n+1$ problem, the $5\times5\times5$ Tic-Tac-Toe, and Hex? They all have extremely simple rules, which unexpectedly lead to chaos: exhibiting unpredictable behavior, without any clear order, without any pattern. These three problems form a good sample, representing a large part (perhaps even the majority) of the applied world problems. Mathematics gave up on these kinds of problems, sending them to the dump called "combinatorial chaos." Is there an escape from the combinatorial chaos?

It is safe to say that understanding/handling combinatorial chaos is one of the main problems of modern mathematics. However, the two game classes (n^d Tic-Tac-Toe and $n\times n$ Hex) represent a bigger challenge, they are even more hopeless, than the $3n+1$ problem. For the $3n+1$ problem we can at least carry out computer experimentation; for example, it is known that the conjecture is true for every $n\le 10^{16}$ (a huge data bank is available): we can search the millions of solved cases for hidden patterns; we can try to extrapolate (which, unfortunately, has not led us anywhere yet).

For the game classes, on the other hand, only a half-dozen cases are solved. Computers do not help: it is easy to simulate a *random play*, but it is impossible to simulate an *optimal play* – this hopelessness leaves the games alive for competition. We simply have no data available; it is impossible to search for patterns if there are no data. (For example, we know only two(!) explicit winning strategies in the whole class of $n\times n\times\cdots\times n=n^d$ Tic-Tac-Toe games: the 3^3 version, which has an easy winning strategy, and the 4^3 version, which has an extremely complicated winning strategy.) These Combinatorial Games represent a humiliating challenge for mathematics!

Note that the subject of Game Theory was created by the Hungarian–American mathematician John von Neumann in a pioneering paper from 1928 and in the well-known book *Theory of Games and Economic Behavior* jointly written with the economist Oscar Morgenstern in 1944. By the way, the main motivation of von Neumann was to understand the role of bluffing in Poker. (von Neumann didn't care, or at least had nothing to say, about combinatorial chaos; the von Neumann–Morgenstern book completely avoids the subject!) Poker is a card game of incomplete information: the game is interesting because neither player knows the opponent's cards. In 1928 von Neumann proved his famous minimax theorem, stating that in games of incomplete information either player has an optimal strategy. This optimal strategy is typically a randomized ("mixed") strategy (to make up for the lack of information).

Traditional Game Theory doesn't say much about games of complete information like Chess, Go, Checkers, and grown-up versions of Tic-Tac-Toe; this is the subject of Combinatorial Game Theory. So far Combinatorial Game Theory has developed in two directions:

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

(I) the theory of "Nim-like games," which means games that fall apart into simple subgames in the course of a play, and
(II) the theory of "Tic-Tac-Toe-like games," which is about games that do not fall apart, but remain a coherent entity during the course of a play.

Direction (I) is discussed in the first volume of the well-known book *Winning Ways* by Berlekamp, Conway, and Guy from 1982. Direction (II) is discussed in this book.

As I said before, the main challenge of Combinatorial Game Theory is to handle combinatorial chaos. To analyze a position in a game (say, in Chess), it is important to examine the options, and all the options of the options, and all the options of the options of the options, and so on. This explains the exponential nature of the game tree, and any intensive case study is clearly impractical even for very simple games, like the $5 \times 5 \times 5$ Tic-Tac-Toe. There are dozens of similar games, where there is a clearcut natural conjecture about which player has a winning strategy, but the proof is hopelessly out of reach (for example, 5-in-a-row in the plane, the status of "Snaky" in Animal Tic-Tac-Toe, Kaplansky's 4-in-a-line game, Hex in a board of size at least 8×8, and so on, see Section 4).

Direction (I), "Nim-like games," basically avoids the challenge of chaos by restricting itself to games with simple components, where an "addition theory" can work. Direction (II) is a desperate attempt to handle combinatorial chaos.

The first challenge of direction (II) is to pinpoint the reasons why these games are hopeless. Chess, Tic-Tac-Toe and its variants, Hex, and the rest are all "Who-does-it-first?" games (which player gives the first checkmate, who gets the first 3-in-a-row, etc.). "Who-does-it-first?" reflects competition, a key ingredient of game playing, but it is not the most fundamental question. The most fundamental question is "What are the achievable configurations, achievable, but not necessarily first?" and the complementary question "What are the impossible configurations?" Drawing the line between "doable" and "impossible" (doable, but not necessarily first!) is the primary task of direction (II). First we have to clearly understand "what is doable"; "what is doable first" is a secondary question. "Doing-it-first" is the ordinary win concept; it is reasonable, therefore, to call "doing it, but not necessarily first" a Weak Win. If a player fails to achieve a Weak Win, we say the opponent forced (at least) a Strong Draw.

The first idea is to switch from ordinary win to Weak Win; the second idea of direction (II) is to carefully define its subject: "generalized Tic-Tac-Toe." Why "generalized Tic-Tac-Toe"? "Tic-Tac-Toe-like games" are the simplest case in the sense that they are static games. Unlike Chess, Go, and Checkers, where the players repeatedly relocate or even remove pieces from the board ("dynamic games"), in Tic-Tac-Toe and Hex the players make permanent marks on the board, and

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

relocating or removing a mark is illegal. (Chess is particularly complicated. There are 6 types of pieces: King, Queen, Bishop, Knight, Rook, Pawn, and each one has its own set of rules of "how to move the piece." The instructions of playing Tic-Tac-Toe is just a couple of lines, but the "instructions of playing Chess" is several pages long.) The "relative" simplicity of games such as "Tic-Tac-Toe" makes them ideal candidates for a mathematical theory.

What does "generalized Tic-Tac-Toe" mean? Nobody knows what "generalized Chess" or "generalized Go" are supposed to mean, but (almost) everybody would agree on what "generalized Tic-Tac-Toe" should mean. In Tic-Tac-Toe the "board" is a $3 \times 3 = 9$ element set, and there are 8 "winning triplets." Similarly, "generalized Tic-Tac-Toe" can be played on an arbitrary finite hypergraph, where the hyperedges are called "winning sets," the union set is the "board," the players alternately occupy elements of the "board." Ordinary win means that a player can occupy a whole "winning set" first; Weak Win simply means to occupy a whole winning set, but not necessarily first.

How can direction (II) deal with combinatorial chaos? The exhaustive search through the exponentially large game-tree takes an *enormous* amount of time (usually more than the age of the universe). A desperate(!) attempt to make up for the lack of time is to study the *random walk* on the game-tree; that is, to study the *randomized game* where both players play randomly.

The extremely surprising message of direction (II) is that the probabilistic analysis of the randomized game can often be *converted* into optimal Weak Win and Strong Draw strategies via potential arguments. It is basically a game-theoretic adaptation of the so-called Probabilistic Method in Combinatorics ("Erdős Theory"); this is why we refer to it as a "fake probabilistic method."

The fake probabilistic method is considered a mathematical paradox. It is a "paradox" because Game Theory is about *perfect* players, and it is shocking that a play between *random generators* ("dumb players") has anything to do with a play between perfect players! "Poker and randomness" is a natural combination: mixed strategy (i.e. random choice among deterministic strategies) is necessary to make up for the lack of complete information. On the other hand, "Tic-Tac-Toe and randomness" sounds like a mismatch. To explain the connection between "Tic-Tac-Toe" and "randomness" requires a longer analysis.

First note that the connection is not trivial in the sense that an optimal strategy is never a "random play." In fact, a "random play" usually leads to a quick, catastrophic defeat. It is a simple general fact that for games of "complete information" the optimal strategies are always deterministic ("pure"). The fake probabilistic method is employed to *find* an explicit deterministic optimal strategy. This is where the connection is: the fake probabilistic method is *motivated* by traditional Probability Theory, but eventually it is *derandomized* by *potential arguments*. In other words, we eventually get rid of Probability Theory completely, but the intermediate

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

"probabilistic step" is an absolutely crucial, inevitable part of the understanding process.

The fake probabilistic method consists of the following main chapters:

(i) game-theoretic first moment,
(ii) game-theoretic second and higher moments,
(iii) game-theoretic independence.

By using the fake probabilistic method, we can find the *exact* solution of infinitely many natural "Ramseyish" games, thought to be completely hopeless before, like some Clique Games, 2-dimensional van der Waerden games, and some "subspace" versions of multi-dimensional Tic-Tac-Toe (the goal sets are at least "2-dimensional").

As said before, nobody knows how to win a "who-does-it-first game." We have much more luck with Weak Win where "doing it first" is ignored. A Weak Win Game, or simply a Weak Game, is played on an arbitrary finite hypergraph, the two players are called Maker and Breaker (alternative names are Builder and Blocker). To achieve an ordinary win a player has to "build and block" at the same time. In a Weak Game these two jobs are separated, which makes the analysis somewhat *easier*, but not *easy*. For example, the notoriously difficult Hex is clearly equivalent to a Weak Game, but it doesn't help to find an explicit first player's winning strategy.

What we have been discussing so far was the achievement version. The Reverse Game (meaning the avoidance version) is equally interesting, or perhaps even more interesting.

The general definition of the *Reverse Weak Game* goes as follows. As usual, it is played on an arbitrary finite hypergraph. One player is a kind of "anti-builder": he wants to avoid occupying a whole winning set – we call him Avoider. The other player is a kind of "anti-blocker": he wants to force the reluctant Avoider to build a winning set – "anti-blocker" is officially called Forcer.

Why "Ramseyish" games? Well, Ramsey Theory gives some partial information about ordinary win. We have a chance, therefore, to compare what we know about ordinary win with that of Weak Win.

The first step in the fake probabilistic method is to describe the majority play, and then, in the second step, try to find a connection between the majority play and the optimal play (the surprising part is that it works!).

The best way to illustrate this is to study the Weak and Reverse Weak versions of the (K_n, K_q) Clique Game: the players alternately take new edges of the complete graph K_n; Maker's goal is to occupy a large clique K_q; Breaker wants to stop Maker. In the Reverse Game, Forcer wants to force the reluctant Avoider to occupy a K_q.

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

If $q = q(n)$ is "very small" in terms of n, then Maker (or Forcer) can easily win. On the other hand, if $q = q(n)$ is "not so small" in terms of n, then Breaker (or Avoider) can easily win. Where is the game-theoretic breaking point? We call the breaking point the Clique Achievement (Avoidance) Number.

For "small" ns no one knows the answer, but for "large" ns we know the exact value of the breaking point! Indeed, assume that n is sufficiently large like $n \geq 2^{10^{10}}$. If we take the lower integral part

$$q = \lfloor 2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 \rfloor$$

(base 2 logarithm), then Maker (or Forcer) wins. On the other hand, if we take the upper integral part

$$q = \lceil 2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 \rceil,$$

then Breaker (or Avoider) wins.

For example, if $n = 2^{10^{10}}$, then

$$2\log_2 n - 2\log_2 \log_2 n + 2\log_2 e - 3 =$$

$$= 2 \cdot 10^{10} - 66.4385 + 2.8854 - 3 = 19,999,999,933.446,$$

and so the largest clique size that Maker can build (Forcer can force Avoider to build) is $19,999,999,933$.

This level of accuracy is even more striking because for smaller values of n we do not know the Clique Achievement Number. For example, if $n = 20$, then it can be either 4 or 5 or 6 (which one?); if $n = 100$, then it can be either 5 or 6 or 7 or 8 or 9 (which one?); if $n = 2^{100}$, then it can be either 99 or 100 or 101 or … or 188 (which one?), that is there are 90 possible candidates. (Even less is known about the small Avoidance Numbers.) We will (probably!) never know the exact values of these game numbers for $n = 20$, or for $n = 100$, or for $n = 2^{100}$, but we know the exact value for a monster number such as $n = 2^{10^{10}}$. This is truly surprising! This is the complete opposite of the usual induction way of discovering patterns from the small cases (the method of direction (I)).

The explanation for this unusual phenomenon comes from our technique: the fake probabilistic method. Probability Theory is a collections of Laws of Large Numbers. Converting the probabilistic arguments into a potential strategy leads to certain "error terms"; these "error terms" become negligible compared to the "main term" if the board is large.

It is also very surprising that the Weak Clique Game and the *Reverse* Weak Clique Game have *exactly* the same breaking point: Clique Achievement Number = Clique Avoidance Number. This contradicts common sense. We would expect that an eager Maker in the "straight" game has a good chance to build a larger clique than a reluctant Avoider in the Reverse version, but this "natural" expectation turns out

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

to be wrong. We cannot give any *a priori* reason why the two breaking points coincide. All that can be said is that the highly technical proof of the "straight" case (around 30 pages) can be easily adapted (like *maximum* is replaced by *minimum*) to yield the same breaking point for the Reverse Game, but this is hardly the answer that we are looking for.

What is the mysterious expression $2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3$? An expert of the theory of Random Graphs immediately recognizes that $2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3$ is exactly 2 less than the Clique Number of the symmetric Random Graph $\mathbf{R}(K_n, 1/2)$ (1/2 is the edge probability).

A combination of the first and second moment methods (standard Probability Theory) shows that the Clique Number $\omega(\mathbf{R}(K_n, 1/2))$ of the Random Graph has a very strong concentration. Typically it is concentrated on a *single* integer with probability $\to 1$ as $n \to \infty$ (and even in the worst case there are at most two values). Indeed, the expected number of q-cliques in $\mathbf{R}(K_n, 1/2)$ equals

$$f(q) = f_n(q) = \binom{n}{q} 2^{-\binom{q}{2}}.$$

The function $f(q)$ drops under 1 around $q \approx 2\log_2 n$. The real solution of the equation $f(q) = 1$ is

$$q = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 1 + o(1), \tag{1}$$

which is exactly 2 more than the game-theoretic breaking point

$$q = 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 3 + o(1) \tag{2}$$

mentioned above.

To build a clique K_q of size (1) by Maker (or Avoider in the Reverse Game) on the board K_n is the majority outcome. The majority play outcome differs from the optimal play outcome by a mere additive constant 2.

The strong concentration of the Clique Number of the Random Graph is not that terribly surprising as it seems at first sight. Indeed, $f(q)$ is a very rapidly changing function

$$\frac{f(q)}{f(q+1)} = \frac{q+1}{n-q} 2^q = n^{1+o(1)}$$

if $q \approx 2\log_2 n$. On an intuitive level, it is explained by the obvious fact that if q switches to $q+1$, then $\binom{q}{2}$ switches to $\binom{q+1}{2} = \binom{q}{2} + q$, which is a large "square-root size" increase.

Is there a "reasonable" variant of the Clique Game for which the breaking point is exactly (1), i.e. the Clique Number of the Random Graph? The answer is "yes," and the game is a "Picker–Chooser game." To motivate the "Picker–Chooser game," note that the alternating Tic-Tac-Toe-like play splits the board into two equal (or almost equal) parts. But there are many other ways to divide the board into two

© Cambridge University Press www.cambridge.org

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

equal parts. The "I-cut-you'll-choose way" (motivated by how a couple shares a single piece of cake after dinner) goes as follows: in each move, Picker picks two previously unselected points of the board, Chooser chooses one of them, and the other one goes back to Picker. In the *Picker–Chooser* game Picker is the builder (i.e. he wants to occupy a whole winning set) and Chooser is the blocker (i.e. his goal is to mark every winning set).

When Chooser is the builder and Picker is the blocker, we call it the *Chooser–Picker* game.

The proof of the theorem that the "majority clique number" (1) is the exact value of the breaking point for the (K_n, K_q) Picker–Chooser Clique Game (where of course the "points" are the edges of K_n) is based on the concepts of:

(a) game-theoretic first moment; and
(b) game-theoretic second moment.

The proof is far from trivial, but not so terribly difficult either (because Picker has so much control of the game). It is a perfect stepping stone before conquering the much more challenging Weak and Reverse Weak, and also the Chooser–Picker versions. The last three Clique Games all have the *same* breaking point, namely (2). What is (2)?

Well, (2) is the real solution of the equation

$$\binom{n}{q}2^{-\binom{q}{2}} = f(q) = \frac{\binom{n}{2}}{2\binom{q}{2}}. \tag{3}$$

The intuitive meaning of (3) is that the overwhelming majority of the edges of the random graph are covered by exactly one copy of K_q. In other words, the Random Graph may have a large number of copies of K_q, but they are well-spread (uncrowded); in fact, there is room enough to be typically pairwise edge-disjoint. This suggests the following *intuition*. Assume that we are at a "last stage" of playing a Clique Game where Maker (playing the Weak Game) has a large number of "almost complete" K_qs: "almost complete" in the sense that, (a) in each "almost complete" K_q all but *two edges* are occupied by Maker, (b) all of these edge-pairs are unoccupied yet, and (c) these extremely dangerous K_qs are pairwise edge-disjoint. If (a)–(b)–(c) hold, then Breaker can still escape from losing: he can block these disjoint unoccupied edge-pairs by a simple Pairing Strategy! It is exactly the Pairing Strategy that distinguishes the Picker–Chooser game from the rest of the bunch. Indeed, in each of the Weak, Reverse Weak, and Chooser–Picker games, "blocker" can easily *win* the Disjoint Game (meaning the trivial game where the winning sets are disjoint and contain at least two elements each) by employing a Pairing Strategy. In sharp contrast, in the Picker–Chooser version Chooser always loses a "sufficiently large" Disjoint Game (more precisely, if there are at least 2^n disjoint n-element winning sets, then Picker wins the Picker–Chooser game).

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

This is the best intuitive explanation that we know to understand breaking point (2). This intuition requests the "Random Graph heuristic," i.e., to (artificially!) introduce a random structure in order to understand a deterministic game of complete information.

But the connection is much deeper than that. To *prove* that (2) is the exact value of the game-theoretic breaking point, one requires a fake probabilistic method. The main steps of the proof are:

(i) game-theoretic first moment,
(ii) game-theoretic higher moments (involving "self-improving potentials"), and
(iii) game-theoretic independence.

Developing (i)–(iii) is a long and difficult task. The word "fake" in the fake probabilistic method refers to the fact that, when an optimal strategy is actually defined, the "probabilistic part" completely disappears. It is a metamorphosis: as a caterpillar turns into a butterfly, the probabilistic arguments are similarly converted into (deterministic) potential arguments.

Note that potential arguments are widely used in puzzles ("one-player games"). A well-known example is Conway's *Solitaire Army* puzzle: arrange men behind a line and then by playing "jump and remove", horizontally or vertically, move a man as far across the line as possible. Conway's beautiful "golden ratio" proof, a striking potential argument, shows that it is *impossible* to send a man forward 5 (4 is possible). Conway's result is from the early 1960s. (It is worthwhile to mention the new result that if "to jump a man diagonally" is permitted, then 5 is replaced by 9; in other words, it is impossible to send a man forward 9, but 8 is possible. The proof is similar, but the details are substantially more complicated.)

It is quite natural to use potential arguments to describe *impossible configurations* (as Conway did). It is more surprising that potential arguments are equally useful to describe *achievable configurations* (i.e. Maker's Weak Win) as well. But the biggest surprise of all is that the Maker's Building Criterions and the Breaker's Blocking Criterions often coincide, yielding *exact* solutions of several seemingly hopeless Ramseyish games. There is, however, a fundamental difference: Conway's argument works for small values such as 5, but the fake probabilistic method gives sharp results only for "large values" of the parameters (we refer to this mysterious phenomenon as a "game-theoretic law of large numbers").

These exact solutions all depend on the concept of "game-theoretic independence" – another striking connection with Probability Theory. What is game-theoretic independence? There is a trivial and a non-trivial interpretation of game-theoretic independence.

The "trivial" (but still very useful) interpretation is about *disjoint* games. Consider a set of hypergraphs with the property that, in each one, Breaker (as the second

© Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-0-521-46100-9 - Combinatorial Games: Tic-Tac-Toe Theory
Jozsef Beck
Excerpt
More information

player) has a strategy to block (mark) every winning set. If the hypergraphs are pairwise disjoint (in the strong sense that the "boards" are disjoint), then, of course, Breaker can block the union hypergraph as well. Disjointness guarantees that in any component either player can play independently from the rest of the components. For example, the concept of the pairing strategy is based on this simple observation.

In the "non-trivial" interpretation, the initial game does *not* fall apart into disjoint components. Instead Breaker can *force* that eventually, in a much later stage of the play, the family of unblocked (yet) hyperedges does fall apart into much smaller (disjoint) components. This is how Breaker can eventually finish the job of blocking the whole initial hypergraph, namely "blocking componentwise" in the "small" components.

A convincing probabilistic intuition behind the non-trivial version is the well-known Local Lemma (or Lovász Local Lemma). The Local Lemma is a remarkable probabilistic sieve argument to prove the *existence* of certain very complicated structures that we are unable to construct directly.

A typical application of the Local Lemma goes as follows:

Erdős–Lovász 2-Coloring Theorem (1975). *Let $\mathcal{F} = \{A_1, A_2, A_3, \dots\}$ be an n-uniform hypergraph. Suppose that each A_i intersects at most 2^{n-3} other $A_j \in \mathcal{F}$ ("local size"). Then there is a 2-coloring of the "board" $V = \bigcup_i A_i$ such that no $A_i \in \mathcal{F}$ is monochromatic.*

The conclusion (almost!) means that there exists a *drawing terminal position* (we have cheated a little bit: in a drawing terminal position, the two color classes have equal size). The very surprising message of the Erdős–Lovász 2-Coloring Theorem is that the "global size" of hypergraph $\mathcal{F}$ is irrelevant (it can even be infinite!), only the "local size" matters.

Of course, the existence of a single (or even several) drawing terminal position does *not* guarantee the existence of a *drawing strategy*. But perhaps it is still true that under the Erdős–Lovász condition (or under some similar but slightly weaker local condition), Breaker (or Avoider, or Picker) has a blocking strategy, i.e. he can block every winning set in the Weak (or Reverse Weak, or Chooser–Picker) game on $\mathcal{F}$. We refer to this "blocking draw" as a Strong Draw.

This is a wonderful problem; we call it the Neighborhood Conjecture. Unfortunately, the conjecture is still open in general, in spite of all efforts trying to prove it during the last 25 years.

We know, however, several partial results, which lead to interesting applications. A very important special case, when the conjecture is "nearly proved," is the class of Almost Disjoint hypergraphs: where any two hyperedges have at most one common point. This is certainly the case for "lines," the winning sets of the n^d Tic-Tac-Toe.

What do we know about the multidimensional n^d Tic-Tac-Toe? We know that it is a draw game even if the dimension d is as large as $d = c_1 n^2 / \log n$, i.e. nearly

© Cambridge University Press www.cambridge.org