PART I

Preliminaries

CHAPTER 1

Sets and mappings

1 Sets

The concept of set is fundamental in mathematics. It is not our purpose to present here an axiomatic account of set theory. Instead we shall assume an intuitive understanding of the terms "set" and "belongs to." Informally speaking, we say that a *set* is a collection of objects (or elements).

If S is a set and x is an element of the set S, we say x *belongs to* S, and we write $x \in S$. An element of a set S is also called a member of S. If x does not belong to S, we write $x \notin S$.

Let A and B be sets. We say that A and B are *equal*, written $A = B$, if they consist of the same elements; that is,

$$x \in A \Leftrightarrow x \in B.$$

(The symbol $\Leftrightarrow$ stands for "if and only if.") A set is thus determined by its elements.

A set with a finite number of elements can be exhibited by writing all of its elements between braces and inserting commas between elements. Thus, {1,2,3} denotes the set whose elements are 1, 2, and 3. The order in which the elements are written makes no difference. Thus, {1,2,3} and {2,1,3} denote the same set. Also, repetition of an element has no effect. For example, {1,2,3,2} is the same set as {1,2,3}. Given a set A and a statement $P(x)$, there is a unique set B whose elements are precisely those elements x of A for which $P(x)$ is true. In symbols, we write $B = \{x \in A | P(x)\}$. When the context is clear, we sometimes also write $B = \{x | P(x)\}$.

There are standard symbols for several sets that we frequently deal

with. Some of these are given below. Others will be introduced subsequently as the occasion arises.

N denotes the set of all natural numbers 1,2,3,....
Z is the set of all integers 0,±1,±2,....
Q is the set of all rational numbers – that is, fractions a/b, where a,b are integers and $b \neq 0$.
R is the set of all real numbers.
C is the set of all complex numbers $x + iy$, where x,y are real numbers and $i^2 = -1$.

For any positive integer n, the set $\{1,2,...,n\}$ is denoted by $\mathbf{n}$. A set S is called *finite* if it has no elements, or if its elements can be listed (counted, labeled) by natural numbers 1,2,3,... and the process of listing stops at a certain number, say n. The number n is called the cardinality of S, and we write $|S| = n$. A set whose elements cannot be listed by the natural numbers 1,2,...,n for any n whatsoever is called an *infinite* set.

A set S is said to be *empty* if S has no elements; that is, the statement $x \in S$ is not true for any x. If S and T are both empty sets, then $S = T$, since the condition $x \in S \iff x \in T$ is satisfied because there is no element x in either S or T to which the condition may be applied. (In such a case we say that the condition is satisfied *vacuously*.) Because any two empty sets are equal, there is just one empty set, which is denoted by $\emptyset$. The empty set is also called the *null* set or the *void* set.

Definition. *Let A and B be sets. A is called a* subset *of B if every element of A is an element of B; that is, if*

$$a \in A \Rightarrow a \in B.$$

(The symbol $\Rightarrow$ stands for "implies.")

If A is a subset of B, we write $A \subset B$. (Some authors write $A \subseteq B$.) Further, if A is a subset of B, we also say that B contains (or includes) A, and we write $B \supset A$ (or $B \supseteq A$).

It follows immediately from the definition that A and B are equal if and only if $A \subset B$ and $B \subset A$. Thus, every set is a subset of itself. Moreover, the empty set $\emptyset$ is a subset of every set because the condition $x \in \emptyset \Rightarrow x \in A$ is satisfied vacuously.

If $S \subset A$, but $S \neq A$, then S is a *proper subset* of A written as $S \subsetneq A$.

Definition. *Let A and B be subsets of a set U. The* union *of A and B, written $A \cup B$, is the set*

$$A \cup B = \{x \in U | x \in A \text{ or } x \in B\}.$$

The intersection *of A and B, written* $A \cap B$, *is the set*

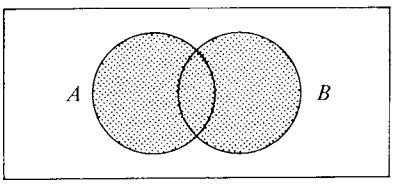
$$A \cap B = \{x \in U | x \in A \text{ and } x \in B\}.$$

The difference *of A and B, written* $A - B$, *is the set*

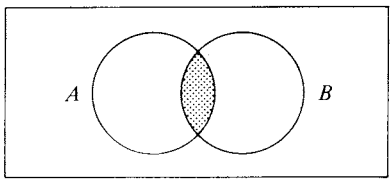
$$A - B = \{x \in U | x \in A \text{ and } x \notin B\}.$$

If $B \subset A$, *then* $A - B$ *is called the* complement *of B in A. A and B are said to be* disjoint *if* $A \cap B$ *is empty* ($A \cap B = \emptyset$).

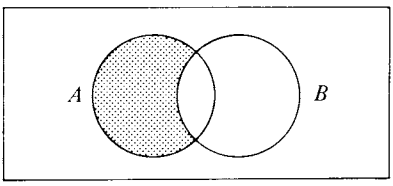
For example, if $A = \{1,2,3\}$ and $B = \{3,4,5\}$, then $A \cup B = \{1,2,3,4,5\}$, $A \cap B = \{3\}$, $A - B = \{1,2\}$, and $B - A = \{4,5\}$.



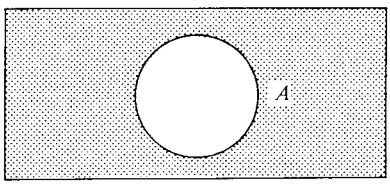
$A \cup B$



$A \cap B$



$A \quad B$



A'

The term *universal set* is sometimes used for a set U that contains all sets in a given context; that is, $X \subset U$ for every set X under consideration. The complement of X in U (namely, the set $U - X$) is then simply called the complement of X and is written X' without explicit reference to U.

It is sometimes helpful to illustrate union, intersection, difference, and complement by means of *Venn diagrams*. We draw circles to represent the given sets A and B and enclose them within a rectangle representing the universal set U. The shaded area in each diagram represents the set $A \cup B$, and so forth, as indicated.

1.1 Theorem. *Let A, B, and C be sets. Then*

(i) $A \cup A = A = A \cap A$.
(ii) $A \cup B = B \cup A$; $A \cap B = B \cap A$.

(iii) $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(v) $A \cup (A \cap B) = A = A \cap (A \cup B)$.

Proof. Left as an exercise. □

1.2 Theorem (DeMorgan's rules). *Let A, B, and X be sets. Then*

(i) $X - (X - A) = X \cap A$.
(ii) $X - (A \cup B) = (X - A) \cap (X - B)$.
(iii) $X - (A \cap B) = (X - A) \cup (X - B)$.

Proof of (i):

$$\begin{aligned} x \in X - (X - A) &\Leftrightarrow x \in X \text{ and } x \notin (X - A) \\ &\Leftrightarrow x \in X \text{ and } x \in A \\ &\Leftrightarrow x \in X \cap A. \end{aligned}$$

Hence, $X - (X - A) = X \cap A$. □

Proof of (ii):

$$\begin{aligned} x \in X - (A \cup B) &\Leftrightarrow x \in X \text{ and } x \notin (A \cup B) \\ &\Leftrightarrow x \in X \text{ and } x \notin A \text{ and } x \notin B \\ &\Leftrightarrow (x \in X \text{ and } x \notin A) \text{ and } (x \in X \text{ and } x \notin B) \\ &\Leftrightarrow x \in X - A \text{ and } x \in X - B \\ &\Leftrightarrow x \in (X - A) \cap (X - B). \end{aligned}$$

Hence, $X - (A \cup B) = (X - A) \cap (X - B)$.
The last part is proved similarly. □

In view of the equality $(A \cup B) \cup C = A \cup (B \cup C)$ (Theorem 1.1), we can do away with the parentheses and simply write $A \cup B \cup C$ to denote unambiguously the union of the sets A, B, and C. Moreover, it is clear that the set $A \cup B \cup C$ consists of all those elements that belong to at least one of the sets A, B, and C. Likewise, $A \cap B \cap C$, the intersection of A, B, and C, is the set of those elements that belong to each of the sets A, B, and C. This suggests the following definition for the union and intersection of an arbitrary number of sets.

Definition. *Let S be a set whose elements are themselves sets. The* union *of all sets in S is defined to be the set*

$$\{x | x \in X \text{ for some } X \text{ in } S\}$$

and is denoted by $\bigcup_{X\in S} X$. *The* intersection *of all sets in S is defined to be the set*

$$\{x | x \in X \text{ for every } X \text{ in } S\}$$

and is denoted by $\bigcap_{X\in S} X$.

If S contains only a finite number of sets, say $X_1,\ldots,X_n$, their union is written $\bigcup_{i=1}^{n} X_i$ or $X_1 \cup \cdots \cup X_n$, and their intersection is written $\bigcap_{i=1}^{n} X_i$ or $X_1 \cap \cdots \cap X_n$.

Definition. *Let X be a set. The set of all subsets of X is called the* power set *of X and is denoted by* $\mathcal{P}(X)$. *That is,*

$$\mathcal{P}(X) = \{S | S \subset X\}.$$

Recall that the empty set $\emptyset$ and the set X itself are subsets of X and are therefore elements of $\mathcal{P}(X)$. For example, let $X = \{1,2\}$. Then the subsets of X are $\emptyset$, $\{1\}$, $\{2\}$, and X. Hence,

$$\mathcal{P}(X) = \{\emptyset,\{1\},\{2\},X\}.$$

If X is the empty set $\emptyset$, then $\mathcal{P}(X)$ has just one element: $\emptyset$.

It should be noted that a and $\{a\}$ are not the same. If $a \in X$, then $\{a\} \in \mathcal{P}(X)$.

1.3 Theorem. *Let X be a finite set having n elements. Then* $\mathcal{P}(X)$ *has* 2^n *subsets. Consequently,* $|\mathcal{P}(X)| = 2^{|X|}$.

Proof. Let us first consider those subsets of X that have r elements each, where $0 \le r \le n$. It is shown in high school algebra that the number of ways in which r elements can be selected out of n elements is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

which is therefore the number of subsets of X having r elements each. Hence, the total number of subsets of X is $\Sigma_{r=0}^{n}\binom{n}{r}$. On putting $a = 1$ in the binomial expansion

$$(1 + a)^n = \sum_{r=0}^{n} \binom{n}{r} a^r,$$

we get

$$\sum_{r=0}^{n} \binom{n}{r} = (1 + 1)^n = 2^n.$$

This proves that X has exactly 2^n subsets. Hence, $\mathcal{P}(X)$ has 2^n elements. Since $n = |X|$, we get $|\mathcal{P}(X)| = 2^{|X|}$. □

Incidentally, the equality $|\mathcal{P}(X)| = 2^{|X|}$ explains why the set of all subsets of X is called the power set of X.

Definition. *Let X be a set. Let π be a set whose elements are nonempty subsets of X; that is, $\pi \subset \mathcal{P}(X)$ and $\emptyset \notin \pi$. If the elements of π are pairwise disjoint and their union is X, then π is called a* partition *of X, and the elements of π are called* blocks *of the partition π.*

For example, the set $\pi = \{\{1,2\},\{3\},\{4,5\}\}$ is a partition of the set $X = \{1,2,3,4,5\}$.

The next theorem follows immediately from the definition.

1.4 Theorem. *Let X be a set. Let π be a set whose elements are nonempty subsets of X. Then π is a partition of X if and only if each element of X belongs to exactly one element of π.*

Proof. Exercise. □

Note that if X is empty, it has only the partition $\pi = \emptyset$.

Problems

1. Prove Theorem 1.1.
2. Prove Theorem 1.2(iii).
3. If a set A has m elements and a set B has n elements, find the number of elements in $A \cup B$. Assume that $A \cap B$ has k elements.
4. After the registration of 100 freshmen, the following statistics were revealed: 60 were taking English, 44 were taking physics, 30 were taking French, 15 were taking physics and French, 6 were taking both English and physics but not French, 24 were taking English and French, and 10 were taking all three subjects.
   (a) Show that 54 were enrolled in only one of the three subjects.
   (b) Show that 35 were enrolled in at least two of them.
5. During quality control checking of a sample of 1000 TV sets, it was found that 100 sets had a defective picture tube, 75 sets had a defective sound system, 80 sets had a defective remote control system, 20 sets had a defective picture tube and a defective remote control, 30 sets had a defective picture tube and a defective sound system, 15 sets had a defective sound system and a defec-

tive remote control system, and 5 sets had all three defects. Use Venn diagrams to show that

(a) 195 sets had at least one defect.
(b) 805 sets had no defects.
(c) 55 sets had a defective picture tube only.
(d) 35 sets had a defective sound system only.
(e) 50 sets had a defective remote control only.

2 Relations

Definition. *Let a,b be elements of a set S. Then the set $\{\{a\},\{a,b\}\}$ is called an* ordered pair *and is denoted by (a,b); a is called the first component (or coordinate), and b is the second component (or coordinate).*

We now show that $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

If $a = c$ and $b = d$, then trivially $(a,b) = (c,d)$. Conversely, let $(a,b) = (c,d)$. Then

$$\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}.$$

By definition of equality of sets, this implies

$$\{a\} = \{c\} \quad \text{or} \quad \{a\} = \{c,d\}.$$

If $\{a\} = \{c\}$, then we must have $\{a,b\} = \{c,d\}$. This yields $a = c$, $b = d$. If, on the other hand, $\{a\} = \{c,d\}$, then we must have $\{a,b\} = \{c\}$. So $a = c = d$ and $a = b = c$, which implies $a = c = b = d$.

Definition. *Let A,B be sets. The set of all ordered pairs (x,y), where $x \in A$ and $y \in B$, is called the* cartesian product *of A and B, in that order, and is denoted by $A \times B$. In symbols,*

$$A \times B = \{(x,y) | x \in A,\ y \in B\}.$$

For example, if $A = \{1,2\}$ and $B = \{a,b,c\}$ then $A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$.

The term "cartesian" is borrowed from coordinate geometry, where a point in the plane is represented by an ordered pair of real numbers (x,y) called its cartesian coordinates. The cartesian product $\mathbf{R} \times \mathbf{R}$ is then the set of cartesian coordinates of all points in the plane.

Definition. *Let A and B be sets, and let R be a subset of $A \times B$. Then R is called a* relation *from A to B. If $(x,y) \in R$, then x is said to be in relation R to y, written $x\ R\ y$. A relation from A to A is called a relation on A (or in A).*

Strictly speaking, a relation is determined by three sets, A,B and a subset R of $A \times B$, although we call it simply the relation R. If R is a relation from A to B, and S is a relation from C to D, then R and S are *equal* if $A = C$, $B = D$, and, for all $x \in A$, $y \in B$, $x\ R\ y \Leftrightarrow x\ S\ y$.

Definition. *Let R be a relation in the set X. R is said to be*

(a) reflexive *if $x\ R\ x$ for all $x \in X$;*
(b) symmetric *if $x\ R\ y$ implies $y\ R\ x$ for all $x,y \in X$;*
(c) antisymmetric *if $x\ R\ y$ and $y\ R\ x$ imply $x = y$ for all $x,y \in X$;*
(d) transitive *if $x\ R\ y$ and $y\ R\ z$ imply $x\ R\ z$ for all $x,y,z \in X$.*

If R is reflexive, symmetric, and transitive, then R is called an equivalence relation *on X. If R is reflexive, antisymmetric, and transitive, then R is called a* partial order *on X.*

2.1 Examples

(a) Let X be the set of all lines in a plane. For $x,y \in X$ let $x \| y$ mean that x is parallel to y. Let us further agree that every line is parallel to itself. Then $\|$ is an equivalence relation on X. Similarly, congruence of triangles and similarity of triangles are equivalence relations.

(b) Let X be a set whose elements are themselves sets. Consider the relation $\subset$ determined by "set inclusion." For any sets $A,B,C \in X$ we see that

(i) $A \subset A$;
(ii) if $A \subset B$ and $B \subset A$, then $A = B$;
(iii) if $A \subset B$ and $B \subset C$, then $A \subset C$.

Hence, set inclusion is reflexive, antisymmetric, and transitive; therefore it is a partial order on X.

(c) The relation $\leq$ ("less than or equal to") on the set $\mathbf{R}$ of real numbers is reflexive, antisymmetric, and transitive; therefore, it is a partial order on $\mathbf{R}$.

(d) The relation *congruence modulo n* on $\mathbf{Z}$ is defined as follows. Let n be a fixed positive integer. For any $x,y \in \mathbf{Z}$, x is said to be *congruent* to y (modulo n), written

$$x \equiv y \pmod{n},$$

if n divides $x - y$. Now, for any x,y,z in $\mathbf{Z}$, it is true that

(i) n divides $x - x = 0$; hence, $x = x \pmod{n}$;
(ii) if n divides $x - y$, then n divides $y - x$;
(iii) if n divides $x - y$ and also $y - z$, then n divides $x - z$.

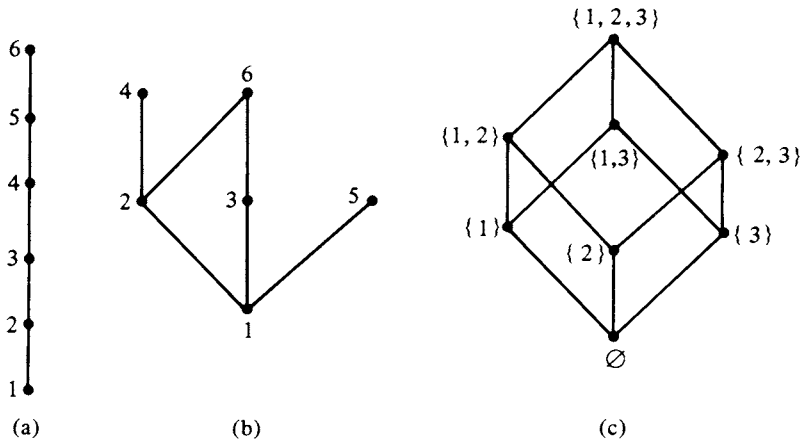
This proves that congruence modulo n is an equivalence relation on $\mathbf{Z}$.

Let X be a set, and let $\leq$ be a partial order on X. (The symbol $\leq$ here denotes an arbitrary partial order and does not necessarily have its usual meaning of "less than or equal to" in real numbers.) The set X together with the partial order $\leq$ is called a *partially ordered set* or, briefly, a *poset*. We refer to it as the poset $(X,\leq)$ or simply the poset X.

Let $(X,\leq)$ be a poset, and let $x,y \in X$. If $x \leq y$, then x is said to be *contained* in y. If $x \leq y$ and $x \neq y$, then x is said to be *properly contained* in y, written $x < y$. If $x < y$ and there is no element a in X such that $x < a < y$, then y is said to *cover* x.

A finite poset X can be represented by a diagram in the following manner. Represent each element in X by a small circle (or a point) in such a way that whenever $x < y$, then y is higher than x in the diagram. Further, join x and y by a straight segment whenever y covers x. As an illustration, we give below the diagrams for the following three posets:

(a) $\{1,2,3,4,5,6\}$ ordered by the usual relation of "less than or equal to";
(b) $\{1,2,3,4,5,6\}$ ordered by divisibility;
(c) $\mathcal{P}(\{1,2,3\})$ ordered by set inclusion.



Let $(X, \leq)$ be a partially ordered set. Let S be a subset of X. An *upper bound* of S is an element $b \in X$ such that $x \leq b$ for all $x \in S$. A *least upper bound* (l.u.b.) of S is an element $m \in X$ such that (i) $x \leq m$ for all $x \in S$ and (ii) if $x \leq m'$ for all $x \in S$ then $m \leq m'$. A *lower bound* and a *greatest lower bound* (g.l.b.) are defined analogously.

It can be easily shown that a l.u.b. (g.l.b.) of S, if it exists, is unique.