CAMBRIDGE TRACTS IN MATHEMATICS

General Editors
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110 **Multivalent Functions**

MULTIVALENT FUNCTIONS

Second edition

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521460262

First published 1958
Second edition 1994

A catalogue record for this publication is available from the British Library

ISBN-13 978-0-521-46026-2 hardback
ISBN-10 0-521-46026-3 hardback

Transferred to digital printing 2006

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Preface

Suppose that we are given a function $f(z)$ regular in the unit circle, and that the equation $f(z) = w$ has there

(a) never more than one solution;
(b) never more than p solutions; or
(c) at most p solutions in some average sense,

as w moves over the open plane. Then $f(z)$ is respectively univalent, p–valent or mean p–valent in $|z| < 1$.

It is the aim of this book to study what we can say about the growth of such functions $f(z)$ and, in particular, to obtain bounds for the modulus and coefficients of $f(z)$ and related quantities. Thus our aim is entirely quantitative in character.

The univalent functions represent the classical case of this theory, and we shall study them in Chapters 1, 7 and 8. By and large the methods of these chapters do not generalize to p–valent or mean p–valent functions. The latter two are studied in Chapters 2, 3, 5 and 6. The theory of symmetrization is developed in Chapter 4, both for its applications to Chapter 5 and for its intrinsic interest. This chapter could reasonably be read by itself. Chapter 7 could be read immediately after Chapter 1 by the student interested mainly in univalent functions. Otherwise the chapters depend on preceding work.

The majority of the material here collected has not, to my knowledge, appeared in book form before, and some of it is quite new. I am, however, extremely indebted to G. M. Golusin's tract [1947] for the contents of each of Chapters 1 and 7. Montel [1933] should also be mentioned, though his approach is rather different from mine. In a tract of this size it is not possible to be exhaustive. Thus I have not been able to find space for Schiffer's variational method, nor Jenkins' theory of modules, both of which have recently scored fine successes in the general field of this book, but I have tried to give references to these results as far as possible. The variational method is developed in Schaeffer and Spencer [1950] and Jenkins has covered his theory in a tract [1958] in the Ergebnisse Series.

The book does demand certain previous knowledge of function theory. Most of this would be contained in the undergraduate course as given, for instance, in Cambridge. When something further is required I have tried to give references to Ahlfors [1979, C. A.] or Titchmarsh [1939] where the results in question can be found. Apart from such references it has been my aim to give detailed proofs of all the theorems. In several cases there is a rather difficult key theorem, from which a number of applications follow fairly simply. In such a case, the reader may omit the proof of the basic theorem on a first reading, until convinced of its value by the application.

Finally I should like to thank all those persons who have helped me with this book, and in particular, Professor Kennedy, Dr. Smithies, Dr. Kövari, Dr. Clunie and Mr. Axtell for much patient criticism in the proof stage and earlier, and Mr. Barry, who kindly prepared the index for me. I am also grateful to the editors for allowing me to publish this book in the Cambridge Tracts series, and to the Cambridge University Press for their patience and helpfulness during all the stages of the preparation of this work.

W. K. H.

LONDON
January 1958

Preface to the Second Edition

In the last 35 years the subject of multivalent and particularly univalent functions has developed rapidly so that it seemed necessary to expand this book considerably. In my choice of new material I have tried to concentrate particularly on fundamental results that are not contained in the two most important books on the subject, that by Pommerenke [1975] and Duren [1983]. Chapter 6 has been devoted to Lucas' bounds for coefficient differences of mean p–valent functions and some new results by Leung and Dawei Shen. I have included Eke's regularity theorems for the maximum, means and coefficients which make it possible to extend results to areally mean p–valent functions which had previously been obtained only for the much narrower class of circumferentially mean p–valent functions.

The most important event in the area of this book has been the proof by de Branges [1985] of Bieberbach's conjecture. Chapter 8 has been added to deal with this subject. I have also included in Chapter 3 results by Clunie, Pommerenke and Baernstein on the coefficients of univalent functions which are bounded or have restricted maximum modulus. Here we have the unusual phenomenon of a type of theorem where the results for univalent functions are significantly stronger than those for mean univalent functions.

Now that Bieberbach's conjecture is proved an analogous conjecture due to Goodman [1948] for the coefficients of p–valent functions constitutes perhaps the most interesting challenge in the area. It is mentioned at the end of Chapter 5. The conjecture is plausible but looks like being extremely difficult and has been proved only in a few very special cases.

Thus there are two completely new chapters and the other chapters all contain new material. All the chapters now contain examples to test the reader's understanding of the material.

I would like to express my debt to Professor Baernstein for his advice on new material, to Professor Pommerenke and Professor Duren for their stimulating books and to May Ghali and Detta Dickinson for their painstaking work in providing Cambridge University Press and me with a camera-ready manuscript. Any remaining mistakes are entirely my own. David Tranah of CUP has been very helpful and supportive throughout.

W. K. H.

YORK
1993