1

Elementary bounds for univalent functions

1.0 Introduction A domain is an open connected set. A function $f(z)$ regular in a domain D is said to be *univalent* in D, if $w = f(z)$ assumes different values w for different z in D. In this case the equation $f(z) = w$ has at most one root in D for any complex w. Such functions map D (1,1) conformally onto a domain in the w plane.

In this chapter we shall obtain some classical results, which give limits for the growth of functions univalent in the unit disc $|z| < 1$. Most of the rest of this tract will aim at generalizing these theorems by proving corresponding results for p-valent functions, i.e. those for which the equation $f(z) = w$ has at most p roots in D, either for every complex w, or in some average sense as w moves over the plane.

If $f(z) = \sum_0^\infty a_n z^n$ is univalent in $|z| < 1$, then so are $f(z) - a_0$ and $(f(z) - a_0)/a_1$, since $a_1 = f'(0) \neq 0$. In fact if a_1 were zero, $f(z)$ would take all values sufficiently near $w = a_0$ at least twice. We thus study the normalized class $\mathfrak{S}$ of functions

$$w = f(z) = z + a_2 z^2 + \ldots$$

univalent in $|z| < 1$.

The two equivalent basic results here are due to Bieberbach [1916] and state that, if $f(z) \in \mathfrak{S}$,† $|a_2| \leq 2$, and that $f(z)$ assumes every value w such that $|w| < \frac{1}{4}$. This latter theorem had been previously proved with a smaller absolute constant by Koebe [1910]. The results of Bieberbach are best possible. We shall first prove them and then develop some of their main consequences.

1.1 Basic results We have

† This symbolic statement stands for '$f(z)$ belongs to the class $\mathfrak{S}$'.

Theorem 1.1 *Suppose that $f(z) \in \mathfrak{S}$. Then $|a_2| \le 2$, with equality only for the Koebe functions*

$$f_\theta(z) = \frac{z}{(1 - ze^{i\theta})^2} = z + 2z^2 e^{i\theta} + 3z^3 e^{2i\theta} + \dots \tag{1.1}$$

We need the following preliminary result:

Lemma 1.1 *Suppose that $w = f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ is regular in a domain containing $|z| = r$, and that the image of $|z| = r$ by $f(z)$ is a simple closed curve $J(r)$, described once. Then the area $A(r)$ enclosed by $J(r)$ is $\pi \left| \sum_{n=-\infty}^{+\infty} n|a_n|^2 r^{2n} \right|$.*

We write $w = f(re^{i\theta}) = u(\theta) + iv(\theta)$, where

$$u(\theta) = \frac{1}{2} \sum_{-\infty}^{+\infty} [a_n e^{in\theta} + \bar{a}_n e^{-in\theta}] r^n,$$

$$v(\theta) = \frac{1}{2i} \sum_{-\infty}^{+\infty} [a_n e^{in\theta} - \bar{a}_n e^{-in\theta}] r^n.$$

Thus

$$\begin{aligned} A(r) &= \left| \int_0^{2\pi} u \frac{dv}{d\theta} d\theta \right| \\ &= \frac{1}{4} \left| \int_0^{2\pi} \left[\sum_{m=-\infty}^{+\infty} r^m (a_m e^{im\theta} + \bar{a}_m e^{-im\theta}) \right] \right. \\ &\qquad \times \left. \left[\sum_{n=-\infty}^{+\infty} n r^n (a_n e^{in\theta} + \bar{a}_n e^{-in\theta}) \right] d\theta \right| \\ &= \left| \frac{\pi}{2} \sum_{n=-\infty}^{+\infty} [a_n(-n a_{-n} + n r^{2n} \bar{a}_n) + \bar{a}_n (n r^{2n} a_n - n \bar{a}_{-n})] \right| \\ &= \pi \left| \sum_{-\infty}^{+\infty} n |a_n|^2 r^{2n} \right|, \end{aligned}$$

since $\sum n a_n a_{-n} = \sum n \bar{a}_n \bar{a}_{-n} = 0$, as we see on replacing n by $-n$ in the summation. Thus the lemma is proved.

Suppose now that

$$w = f(z) = z + a_2 z^2 + \dots \in \mathfrak{S}.$$

Then so does $F(z) = [f(z^2)]^{\frac{1}{2}} = z + \frac{1}{2} a_2 z^3 + \dots$ In fact $f(z^2)$ does not vanish except at $z = 0$, where it has a double zero, and if $F(z_1) = F(z_2)$, then $f(z_1^2) = f(z_2^2)$, and so $z_1^2 = z_2^2$, i.e. $z_1 = \pm z_2$. But $F(z)$ is an odd function, so that $z_1 = -z_2$ gives $F(z_1) = -F(z_2)$. Hence we must have $z_1 = z_2$. Also since $f(z^2)$ has only a single zero of order two, $F(z)$ is regular. Therefore $F(z)$ is univalent.

Next write

$$g(z) = \frac{1}{F(z)} = \frac{1}{z} - \frac{1}{2}a_2 z + \ldots = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n.$$

Then $g(z)$ is univalent in $0 < |z| < 1$, and so the image of $|z| = r$ by $g(z)$ is a simple closed curve for $0 < r < 1$. Hence by Lemma 1.1

$$-\frac{1}{r^2} + \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = \pm\frac{A(r)}{\pi}$$

does not vanish for $0 < r < 1$. The left-hand side is clearly negative for small positive r and so for $0 < r < 1$. As $r \to 1$ we deduce that

$$\sum_{n=1}^{\infty} n|b_n|^2 \le 1.$$

Thus we have $|b_1| = \frac{1}{2}|a_2| \le 1$, and equality is possible only if $b_n = 0 \quad (n > 1)$, and in this case

$$g(z) = \frac{1}{z} - ze^{i\theta}, \quad F(z) = \frac{z}{1 - z^2 e^{i\theta}}, \quad f(z) = \frac{z}{(1 - ze^{i\theta})^2}.$$

This proves Theorem 1.1.

We deduce immediately

Theorem 1.2 *Suppose that $f(z) \in \mathfrak{S}$ and that $f(z) \ne w$ in $|z| < 1$. Then $|w| \ge \frac{1}{4}$. Equality is possible only if $f(z)$ is given by (1.1) and $w = -\frac{1}{4}e^{-i\theta}$.*

Since $f(z) \ne w$

$$\frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \ldots \in \mathfrak{S}.$$

Thus Theorem 1.1 gives

$$\left|a_2 + \frac{1}{w}\right| \le 2, \quad \left|\frac{1}{w}\right| \le 2 + |a_2| \le 4, \quad |w| \ge \frac{1}{4},$$

as required. Equality is possible only if $a_2 = 2e^{i\theta}$, $w^{-1} = -4e^{i\theta}$, and then $f(z)$ must be given by $f_\theta(z)$ in (1.1).

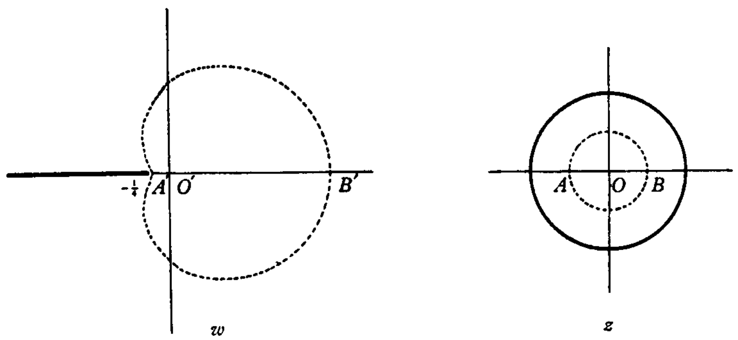

Fig. 1.

We note finally that the function

$$f_0(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

maps $|z| < 1$ (1,1) conformally onto the w plane cut from $-\frac{1}{4}$ to $-\infty$ along the negative real axis. Thus $f_0(z) \in \mathfrak{S}$ and $f_0(z) \neq -\frac{1}{4}$ in $|z| < 1$. Hence the functions $f_\theta(z) = e^{-i\theta} f_0(ze^{i\theta})$ of (1.1) also belong to $\mathfrak{S}$ and $f_\theta(z) \neq -\frac{1}{4}e^{-i\theta}$. Thus Theorems 1.1 and 1.2 are best possible.

We shall see that these functions $f_\theta(z)$ are extreme in $\mathfrak{S}$ for a variety of other problems also.

In 1984 de Branges [1985] solved the outstanding problem in the theory, by proving the Bieberbach conjecture that $|a_n| \leq n$ holds for $f(z) \in \mathfrak{S}$ and $n > 1$ with equality only for $f(z) = f_\theta(z)$. We shall give the proof when $n = 3$, which is due to Löwner [1923] in Chapter 7. In Chapter 8 we shall prove de Branges' Theorem in full generality.

1.2 Elementary growth and distortion theorems We can develop an interesting further group of inequalities as a direct consequence of Theorem 1.1.

Theorem 1.3 *Suppose that $f(z) \in \mathfrak{S}$. Then, for $|z| = r$ $(0 < r < 1)$ we have*†

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \tag{1.2}$$

† Bieberbach [1916], Gronwall [1916], Szegö [1928].

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$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \tag{1.3}$$

$$\frac{1-r}{r(1+r)} \leq \left|\frac{f'(z)}{f(z)}\right| \leq \frac{1+r}{r(1-r)}. \tag{1.4}$$

Equality holds in all cases only for the functions $f_\theta(z)$ of (1.1).

We assume that $|z_0| < 1$, and set

$$\phi(z) = f\left(\frac{z_0 + z}{1 + \bar{z}_0 z}\right) = b_0 + b_1 z + b_2 z^2 + \ldots \tag{1.5}$$

Then clearly $\phi(z)$ is univalent in $|z| < 1$. Further

$$b_0 = f(z_0), \quad b_1 = \phi'(0) = (1 - |z_0|^2) f'(z_0),$$
$$b_2 = \tfrac{1}{2}\phi''(0) = \tfrac{1}{2}(1 - |z_0|^2)^2 f''(z_0) - \bar{z}_0(1 - |z_0|^2) f'(z_0).$$

We apply Theorem 1.1 to $(\phi(z) - b_0)/b_1$ in $\mathfrak{S}$ and obtain $|b_2| \leq 2|b_1|$, i.e.

$$|f''(z_0)(1 - |z_0|^2)^2 - 2\bar{z}_0 f'(z_0)(1 - |z_0|^2)| \leq 4(1 - |z_0|^2)|f'(z_0)|.$$

Writing $z_0 = \rho e^{i\theta}$ we deduce

$$\left| z_0 \frac{f''(z_0)}{f'(z_0)} - \frac{2\rho^2}{1-\rho^2} \right| \leq \frac{4\rho}{1-\rho^2}. \tag{1.6}$$

Since

$$\frac{\partial}{\partial \rho} \log |f'(\rho e^{i\theta})| = \Re e^{i\theta} \frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})},$$

we obtain at once

$$\frac{2\rho - 4}{1-\rho^2} \leq \frac{\partial}{\partial \rho} \log |f'(\rho e^{i\theta})| \leq \frac{2\rho + 4}{1 - \rho^2}.$$

On integrating this from 0 to r with respect to ρ, we deduce (1.3).

We deduce immediately that

$$|f(re^{i\theta})| \leq \int_0^r |f'(\rho e^{i\theta})| d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho = \frac{r}{(1-r)^2},$$

and this is the right-hand inequality in (1.2). To obtain the lower bound for $|f(re^{i\theta})|$, we assume without loss of generality, that $f(re^{i\theta}) = Re^{i\phi}$, where $R < \frac{1}{4}$, since otherwise there is nothing to prove. It then follows from Theorem 1.2 that the straight line segment λ from 0 to $Re^{i\phi}$ lies entirely in the image of $|z| < 1$ by $f(z)$. Hence λ corresponds to a path l in $|z| < 1$, which joins $z = 0$ to $re^{i\theta}$. Thus if $t = |z|$ we deduce from (1.3)

$$R = \int_\lambda |dw| = \int_l \left|\frac{dw}{dz}\right| |dz| \geq \int_l \frac{1-t}{(1+t)^3} dt = \frac{r}{(1+r)^2},$$

and this completes the proof of (1.2).

Finally, we apply (1.2) to $[\phi(z) - b_0]/b_1$, where $\phi(z)$ is defined by (1.5). This gives

$$|b_1| \frac{|z|}{(1+|z|)^2} \leq \left| f\left(\frac{z_0 + z}{1 + \bar{z}_0 z}\right) - f(z_0)\right| \leq |b_1| \frac{|z|}{(1-|z|)^2}.$$

Putting $z = -z_0$, $b_1 = (1 - |z_0|^2) f'(z_0)$, we deduce (1.4).

It is easily seen that the functions $f_\theta(z)$ of (1.1) yield equality in the right-hand inequalities of (1.2)–(1.4) when $z = re^{-i\theta}$, and in the left-hand inequalities if $z = -re^{-i\theta}$. On noting that by Theorem 1.1 equality is possible in (1.6) and hence in the subsequent inequalities only if $\phi(z)$ reduces to one of the functions $f_\theta(z)$, we easily see that no other functions can give equality in Theorem 1.3.

1.2.1 We develop now another proof of some of the results in Theorem 1.3, which is based on Theorem 1.2, rather than 1.1. Once Theorem 1.2 has been extended to more general classes of functions, the present proof will also generalize. To make this evident we introduce the following:

Definition *Let* $f(z) = z + a_2 z^2 + \ldots$ *be regular in* $|z| < 1$. *We shall say that* $f(z) \in \mathfrak{S}_0$ *if, given any complex* z_0, *with* $|z_0| < 1$, *and any function* $\omega(\zeta)$ *univalent and satisfying* $|\omega(\zeta)| < 1$, $\omega(\zeta) \neq z_0$ in $|\zeta| < 1$, *we have for* $\phi(\zeta) = f[\omega(\zeta)]$

$$|\phi'(0)| \leq 4(|\phi(0)| + |f(z_0)|).$$

We note that $\mathfrak{S}$ is a subclass of $\mathfrak{S}_0$. For if $f(z) \in \mathfrak{S}$, $\phi(\zeta) = f[\omega(\zeta)]$ is univalent in $|\zeta| < 1$, and $\phi(\zeta) \neq f(z_0) = w_0$ say. We write $\phi(\zeta) = b_0 + b_1\zeta + \ldots$, and apply Theorem 1.2 to $(\phi(\zeta) - b_0)/b_1$ which belongs to $\mathfrak{S}$ and never takes the value $(w_0 - b_0)/b_1$. Thus

$$\left|\frac{w_0 - b_0}{b_1}\right| \geq \frac{1}{4}; \quad |b_1| = |\phi'(0)| \leq 4|w_0 - b_0| \leq 4[|f(z_0)| + |\phi(0)|],$$

and so $f(z) \in \mathfrak{S}_0$.

We shall see in Chapter 5 that the class $\mathfrak{S}_0$ is effectively a good deal larger than $\mathfrak{S}$, and so the results of Theorems 1.4 and 1.5 will apply to a significantly more general class of functions than the univalent ones.

Theorem 1.4 *Suppose that* $f(z) = z + a_2 z^2 + \ldots \in \mathfrak{S}_0$. *Then*

$$|a_2| \le 2. \tag{1.7}$$

Further, we have for $|z| = r\,(0 < r < 1)$

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \tag{1.8}$$

$$|f'(z)| \le \frac{1+r}{r(1-r)}|f(z)| \le \frac{1+r}{(1-r)^3}. \tag{1.9}$$

Finally, the equation $f(z) = w$ *has exactly one root in* $|z| < 1$ *if* $|w| < \frac{1}{4}$.

In return for our greater generality we have lost only the left inequalities in (1.3) and (1.4). This is inevitable, since the derivatives of functions in $\mathfrak{S}_0$ may well vanish in $|z| < 1$.

To prove Theorem 1.4, put

$$\frac{z}{(1-z)^2} = Z = \frac{4d\zeta}{(1-\zeta)^2},$$

where $d = r/(1+r)^2$ for some fixed r satisfying $0 < r < 1$. Then $|z| < 1$ cut from $-r$ to -1 along the negative real axis is mapped (1,1) conformally onto the Z plane cut from $-d$ to $-\infty$ along the negative real axis and so onto $|\zeta| < 1$. Thus if we write $z = \omega(\zeta)$, $\phi(\zeta) = f[\omega(\zeta)]$, then $\omega(\zeta)$ is univalent, $\omega(\zeta) \ne -r$ in $|\zeta| < 1$, and so, since $f(z) \in \mathfrak{S}_0$, we have

$$|\phi'(0)| \le 4\{|\phi(0)| + |f(-r)|\},$$

i.e.

$$4d|f'(0)| \le 4|f(-r)|.$$

Since $f'(0) = 1$, this gives $|f(-r)| \ge d$, and on applying the argument to $e^{-i\theta} f(ze^{i\theta})$, which belongs to $\mathfrak{S}_0$ if $f(z) \in \mathfrak{S}_0$, we have the left inequality of (1.8).

It follows immediately that $f(z) \ne 0$ in $|z| < 1$ except at $z = 0$. Next it follows from Rouché's Theorem† that if $|w| < r(1+r)^{-2}$, $f(z)$ and $f(z) - w$ have an equal number of zeros in $|z| < r$, i.e. exactly one. Making r tend to 1, we deduce that, for $|w| < \frac{1}{4}$, $f(z) = w$ has exactly one root in $|z| < 1$.

Next choose θ so that $a_2 e^{i\theta} = -|a_2|$. Then as $r \to 0$,

$$|f(re^{i\theta})| = |r + a_2 e^{i\theta} r^2 + O(r^3)| = r - |a_2| r^2 + O(r^3),$$

† See e.g. Ahlfors [1979, hereafter called C.A., p. 153.]

and

$$|f(re^{i\theta})| \geq \frac{r}{(1+r)^2} = r - 2r^2 + O(r^3),$$

by the left inequality of (1.8). This gives $|a_2| \leq 2$.

It remains to prove the inequalities (1.9) and the right inequality of (1.8). We put

$$Z = \frac{z}{(1-z)^2} = k\left(\frac{1+\zeta}{1-\zeta}\right)^2, \text{ where } k = \frac{r}{(1-r)^2}.$$

Here r is a fixed positive number such that $0 < r < 1$. Then $|\zeta| < 1$ corresponds (1,1) conformally to the Z plane cut along the negative real axis and to $|z| < 1$ cut along the real axis from -1 to 0. We again write $z = \omega(\zeta)$, $\phi(\zeta) = f[\omega(\zeta)]$. Then $\omega(\zeta) \neq 0$ in $|\zeta| < 1$, and since $f(0) = 0$, $f(z) \in \mathfrak{S}_0$, we have

$$|\phi'(0)| = \frac{(1-r)^3}{1+r} 4k|f'(r)| \leq 4|\phi(0)| = 4|f(r)|.$$

Since $e^{-i\theta} f(ze^{i\theta}) \in \mathfrak{S}_0$ also, we deduce

$$|f'(re^{i\theta})| \leq \frac{1+r}{k(1-r)^3}|f(re^{i\theta})| = \frac{1+r}{r(1-r)}|f(re^{i\theta})|,$$

and this is the left inequality of (1.9). Hence

$$\frac{\partial}{\partial r} \log |f(re^{i\theta})| \leq \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \leq \frac{1+r}{r(1-r)}, \tag{1.10}$$

and integrating this from r_1 to r_2, where $0 < r_1 < r_2 < 1$, we deduce

$$\log \left|\frac{f(r_2 e^{i\theta})}{f(r_1 e^{i\theta})}\right| \leq \int_{r_1}^{r_2} \frac{(1+r)dr}{r(1-r)} = \log\left[\frac{(1-r_1)^2 r_2}{r_1(1-r_2)^2}\right],$$

or

$$\frac{(1-r_2)^2}{r_2}|f(r_2 e^{i\theta})| \leq \frac{(1-r_1)^2}{r_1}|f(r_1 e^{i\theta})|. \tag{1.11}$$

Making r_1 tend to 0 in this we deduce the right-hand inequalities of (1.8) and (1.9) with $r = r_2$. This completes the proof of Theorem 1.4.

From the point of view of later applications it is worth while to note the following consequences of (1.9):

Theorem 1.5 *Suppose that $f(z) = z + a_2 z^2 + \ldots \in \mathfrak{S}_0$ and set*

$$M(r, f) = \max_{|z|=r} |f(z)| \quad (0 < r < 1).$$

Then, unless $f(z) = f_\theta(z) = z(1 - ze^{i\theta})^{-2}$, $(1-r)^2 r^{-1} M(r,f)$ *decreases strictly with increasing* r $(0 < r < 1)$, *and so tends to* α *as* $r \to 1$, *where* $0 \le \alpha < 1$. *Hence the upper bounds for* $|f(z)|, |f'(z)|$ *given by (1.8) and (1.9) respectively are attained only by the functions* $f_\theta(z)$.

To prove Theorem 1.5 note that equality can hold in (1.11) only if equality holds in both the inequalities of (1.10) for $r_1 < r < r_2$. This gives

$$\Re e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} = \frac{1+r}{r(1-r)},$$

and so

$$\Im e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} = 0 \quad (r_1 < r < r_2),$$

i.e.

$$z\frac{f'(z)}{f(z)} = \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}$$

for $z = re^{i\theta}$ $(r_1 < r < r_2)$ and so, by analytic continuation, throughout $|z| < 1$. In this case $f(z) = f_{-\theta}(z)$.

Otherwise strict inequality holds in (1.11) for $0 < r_1 < r_2 < 1$, and $0 \le \theta \le 2\pi$. Choose θ so that $|f(r_2 e^{i\theta})| = M(r_2, f)$. Then (1.11) gives

$$\frac{(1-r_2)^2}{r_2} M(r_2, f) < \frac{(1-r_1)^2}{r_1} |f(r_1 e^{i\theta})| \le \frac{(1-r_1)^2}{r_1} M(r_1, f).$$

Hence, unless $f(z) = f_\theta(z), \psi(r) = (1-r)^2 r^{-1} M(r,f)$ decreases strictly with increasing r $(0 < r < 1)$, and $\psi(r) \le 1$ by (1.9). Thus $\psi(r) < 1$ $(0 < r < 1)$, so that the upper bounds for $|f(z)|$ in (1.8) and for $|f'(z)|$ in (1.9) are not attained and $\lim_{r\to 1} \psi(r) = \alpha < 1$. This proves Theorem 1.5.

1.3 Means and coefficients We referred above to de Branges' Theorem that $|a_n| \le n$ holds when $f(z) \in \mathfrak{S}$ and $n \ge 2$. We proceed to prove the simpler inequality $|a_n| < en$ due to Littlewood [1925].

Theorem 1.6 *Suppose that* $f(z) = z + a_2 z^2 + \ldots \in \mathfrak{S}$. *Then*

$$I_1(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})| d\theta < \frac{r}{1-r} \quad (0 < r < 1), \tag{1.12}$$

and so

$$|a_n| < eI_1\left[\frac{n-1}{n}, f\right] < en \quad (n \ge 2). \tag{1.13}$$

As we saw when proving Theorem 1.1,

$$\phi(z) = [f(z^2)]^{\frac{1}{2}} = z + b_3 z^3 + b_5 z^5 + \ldots \in \mathfrak{S},$$

and by Theorem 1.3, applied to $f(z)$, we have $|\phi(z)| \le r/(1-r^2)$ for $|z| \le r$. We note that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |\phi'(re^{i\theta})|^2 d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \phi'(re^{i\theta})\overline{\phi'(re^{i\theta})} d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} n b_n r^{n-1} e^{i(n-1)\theta}\right)\left(\sum_{m=1}^{\infty} m\bar{b}_m r^{m-1} e^{-i(m-1)\theta}\right) d\theta \\
&\quad = \sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2n-2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\pi\sum_{n=1}^{\infty} n|b_n|^2 r^{2n} &= \int_0^r \rho d\rho \int_0^{2\pi} |\phi'(\rho e^{i\theta})|^2 d\theta \\
&= \{\text{area of transform of } |z| < r \text{ by } w = \phi(z)\} \\
&< \pi\left(\frac{r}{1-r^2}\right)^2. \qquad (1.14)
\end{aligned}$$

For since $\phi(z)$ is univalent, the area of the transform is at most πR^2, where R is the greatest distance of the transform from $w = 0$.†

Integrating term by term from 0 to r after division by r we obtain

$$\sum_{n=1}^{\infty} |b_n|^2 r^{2n} < \frac{r^2}{1-r^2}.$$

But

$$\begin{aligned}
I_1(r^2, f) &= \frac{1}{2\pi}\int_0^{2\pi} |f(r^2 e^{i\theta})| d\theta = \frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})\overline{\phi(re^{i\theta})}\, d\theta \\
&= \sum_{n=1}^{\infty} |b_n|^2 r^{2n}.
\end{aligned}$$

† Equality is clearly excluded here.