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with applications to algebra and number theory

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Preface

'Funny that you should ask. As it happens I have a complete mathematical vision of power. You could call it a programme. Some pieces are in place, some theorems are proved, others await proof and revelation. It is based on the theory of schemes.'

'Of finite type?' asked Zhilin, with rapt curiosity.

'Any type, my friend, any type at all.'

'Schemes, eh.' Eli nodded his approval.'Good. Good.'

'Schemes are a little like the crania of topology with little light bulbs of algebra, called sheaves, stuck all over them.'

'Sheaves, very good. Very agriculdural.'

from *The Yukiad* (Snaith, 1990c)

This volume began as a one-term advanced graduate course in algebra which I gave at the beginning of 1990 at McMaster University. As originally conceived my plan was to give a brief introduction to the representation theory of finite groups in characteristic zero. This sketch was to have been succeeded by an outline of the topological construction of my original Explicit Brauer Induction formula (Snaith, 1988b, 1989b) followed by a description of the behaviour of Explicit Brauer Induction with respect to Adams operations 4.1.6 as originally proved in theorem 2.33 of Snaith (1989a). Equipped with 4.1.6 the course was then to have concluded with a discussion of class-groups of group-rings and a proof of M.J. Taylor's conjecture concerning determinantal congruences 4.3.10 (see also the stronger congruences of 4.3.37).

However, in 1989, I learnt of the work of Robert Boltje (1989, 1990) which axiomatised Explicit Brauer Induction formulae and, entirely algebraically, found a different formula. This second formula was easier to use than my original one — being a homomorphism rather than a derivation (see 2.3.28) — and its purely algebraic derivation was far better

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suited for exposition in a graduate course on algebra! As it happens, the formulae of Snaith (1988b) and Boltje (1990) are related by an equation which is to be found in 2.5.11. Indeed, for p-groups this relation may be used to derive either formula from the other (see 2.5.16).

By the time I had concluded the course I had collected quite a number of applications of Explicit Brauer Induction, several of which had figured vaguely in my original motivation but had taken sufficiently long to develop that they could not be included in Snaith (1989b). At that point it seemed to me advantageous to have an algebraic treatment of Explicit Brauer Induction, together with a selection of typical applications for the benefit of those who did not like the topological proofs of Snaith (1989b) which were apt, on occasion, to resort to stable homotopy theory to derive algebraic results!

Having described the origins of this book, let me move on to sketch its contents. Each chapter has a more detailed introduction to which the reader is referred for a more complete account. In general, the topics concern examples of 'applied representation theory' in which algebraic objects (class-groups, for example) are studied by means of finite-dimensional, complex representations of finite groups. Under these circumstances Explicit Brauer Induction enables one to use Brauer's induction theorem either *constructively* (see 6.3.6, for example) or with greater control (see 4.3.37). As a result, almost all the main results presented here are either new (for example, 4.3.37, 6.3.6 and 6.3.20) or are proved by a new method (for example, 4.5.39, 4.6.3 and 7.3.56).

Chapter 1 quickly covers the basic material concerning the finite-dimensional representations of a finite group. We specialise almost immediately to the case of complex representations, emphasising the properties of induced representations.

Chapter 2 begins with an account of Brauer's canonical version of Artin's induction theorem. Brauer's induction theorem, in its classical existential (non-canonical) form, is proved topologically by an extension of Snaith's (1988b) method. The formalism of Explicit Brauer Induction is recapitulated from Snaith (1989a,b) and Boltje's axioms are stated and shown to yield an Explicit Brauer Induction homomorphism *with rational coefficients*. The difficult part of the proof is to show that integrality of the Explicit Brauer Induction map, a_G, which is accomplished by Boltje's (1990) argument. However, for completeness, I have included a description of the topological construction of a_G which is due to Peter Symonds (1991) and which provides those who have topological tastes with a more conceptual, alternative point of view. As I have

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remarked previously, natural Explicit Brauer Induction formulae are not unique, since Snaith's (1988b) and Boltje's (1990) formulae are generally different. The chapter concludes by deriving a third Explicit Brauer Induction homomorphism, d_G, which is due to Robert Boltje. The map, d_G, has rational coefficients, is natural and commutes with induction. This remarkable homomorphism was discovered during the course of the joint work, which appears in Chapter 6, by R. Boltje, G-M. Cram and myself on conductors in the non-separable residue field case.

Chapter 3 studies the Explicit Brauer Induction formula when applied to an irreducible representation of the group of invertible, 2×2 matrices with entries in the field with q elements. Each of the irreducible representations of GL_2F_q is constructed and the leading terms of the Explicit Brauer Induction formulae are evaluated. In addition, it is shown how this 'leading term data' enables one to construct the Shintani correspondence between Frobenius-invariant, irreducible representations of one general linear group and the irreducible representations of the subgroup of Frobenius-fixed matrices. Our construction, which makes no mention of character-values or the Shintani norm map, offers an appealingly intrinsic approach to the 'base-change' which indicates just how useful Explicit Brauer Induction might prove to be if it were possible to extend the technique to the case of admissible representations of the general linear group of a local field (Gérardin & Labesse, 1979).

Chapter 4 introduces the Adams operations in the complex representation ring of a finite group (to be precise, in this special case these operations were originally due to Burnside). It is shown how the Explicit Brauer Induction formula enables one to write its image under any Adams operation as a linear combination of monomial representations. Next the Hom-description is given for the class-group of finitely generated, projective modules over the integral group-ring of a finite group. This description presents the class-group as a quotient of idèle-valued functions on the representation ring of the group modulo global-valued functions and other special functions which are called determinants. Martin Taylor (1978) conjectured that the Adams operations, when applied to determinantal functions, would satisfy some congruences modulo the residue degree. This determinantal congruence conjecture is proved, strengthened and reformulated to give some new homomorphisms whereby to detect class-groups. The chapter concludes with new proofs of two results concerning determinants. It is shown that the determinantal subgroup satisfies Galois descent in tamely ramified extensions, which was one of the three mains steps in M.J. Taylor's proof

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of Fröhlich's conjecture (Taylor, 1981). It is also shown that Adams operations preserve the determinantal subgroup, which was originally proved in a different manner by Ph. Cassou-Noguès and M.J. Taylor (Taylor, 1984).

Chapter 5 deals with six topics which are united merely by the fact that they are related to the class-group of a group-ring. In the previous chapter new maps out of the class-group were constructed by means of determinantal congruences. This chapter commences with the construction of *restricted determinants*, which are new maps into the class-group. The construction of the restricted determinant homomorphisms is later refined to yield a technique for the detection of the class-group of a maximal order in the rational group-ring of a finite group. In Chapter 7 this detection technique, which uses some new types of Hom-groups, is applied to give a new proof of David Holland's theorem, which states that the Fröhlich–Chinburg conjecture is true in the class-group of a maximal order. This chapter also contains a calculation of the class-group of the integral group-ring of the quaternion group of order eight. Every text on class-groups has its own version of this calculation; mine is accomplished by means of a little homological algebra and a new type of reduced norm invariant. Two sections deal with the subgroup, called the Swan subgroup, which is generated in the class-group by the projective modules which were so elegantly constructed by R.G. Swan (1960). By topological techniques using groups actions on spheres new families of relations between Swan modules are derived. These relations, together with the algebra of determinantal congruences, are used to calculate the Swan subgroup of some types of nilpotent groups. These calculations prompt us to venture a conjecture as to the identity of the Swan subgroup of any nilpotent group. The final section illustrates the non-triviality of the class defined in the class-group by the roots of unity which reappear in the material on real cyclotomic Galois module structure at the end of Chapter 7.

Chapter 6 deals with the problem, raised in Serre (1960) and Kato (1989), of constructing a Swan conductor for Galois representations of complete, discrete valuation fields whose residue fields are inseparable. The classical theory of the Artin and Swan conductors is recalled, together with Kato's definition of a Swan conductor for a one-dimensional representation. The Explicit Brauer Induction formula is used to define the required Swan conductor in general, and many properties are derived for this new conductor, which coincides with the classical Swan conductor when the residue field extension is separable. Examples are

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given to show that no such generalisation of the Swan conductor can possess the classical property of 'inductivity in dimension zero'. However, using the rational-valued Explicit Brauer Induction homomorphism, d_G, of Chapter 2 we show that our Swan conductor is 'inductive in dimension zero' if it is so on all p-subgroups of a particularly simple type.

Chapter 7 treats the Galois module structure of the unit group in a ring of algebraic integers. The main conjecture in this topic is due to A. Fröhlich and T. Chinburg and concerns the identity in the class-group of the group-ring of a Galois group of certain invariants which are constructed from the theory of class formations. In the tamely ramified case one of these invariants is simply the projective module furnished by the additive group of the ring of integers in the tame Galois extension of number fields. In general, one studies the global Chinburg invariant and attempts to equate it with an analytically defined class due to A. Fröhlich and Ph. Cassou-Noguès. The chapter begins with the construction of the local Chinburg invariants. These are described explicitly by means of the theory of central simple algebras and from this description a new proof is given of the result, due to T. Chinburg (1985), that the local invariants vanish in the tamely ramified case. The global Chinburg invariant is defined and the main conjecture is stated. The chapter contains a new proof, using the method of Chapter 5, of D. Holland's (1992) result, which states that the Fröhlich–Chinburg conjecture is true in the class-group of a maximal order. The chapter closes with some new families of (cyclotomic) extensions for which the Fröhlich–Chinburg conjecture is true in the class-group.

Each chapter is endowed with a selection of exercises, which vary from relatively straightforward problems, such as the completions of omitted proofs, to research problems suggested by the topics treated in the text.

As mentioned above, parts of this book have served as the basis for an advanced graduate course in algebra. In fact the material lends itself to such a purpose in a number of ways and the reader may find the following suggestions helpful in designing such courses. Twice I have given a course which covered Chapters 1 and 2 and then concluded by applying the results on Adams operations in Chapter 4 to derive the determinantal congruences. Such a course was particularly successful, since it assumes only a minimal algebraic background and culminates in a proof of M.J. Taylor's conjecture concerning determinantal congruences — all this being accomplished within 40 hours of lectures. A one-term introductory course on representation theory can be made from Chapter 1, and Chapter 3 and can be given some additional mystique by assuming

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the existence of the Explicit Brauer Induction homomorphism, a_G, in order to analyse the Shintani correspondence in the manner of Chapter 3. Finally, several two-term courses may be designed by combining the contents of the first two chapters with any one of Chapter 4 (and possibly Chapter 5), Chapter 6 or Chapter 7. The last three alternatives are sufficiently advanced that, in the course of the related exercises, they introduce the student to a number of interesting research problems.

In the course of writing this book I have been helped by conversations, correspondence and comments from many mathematicians. In particular, I am very grateful to Robert Boltje for his improvements on my original Explicit Brauer Induction formula and for his help in our joint work with Georg-Martin Cram, which forms Chapter 6. Conversations with Greg Hill, Charles Curtis and my student, Brian McCudden, gave rise to the treatment of Shintani descent which appears in Chapter 2. I am also grateful to Ted Chinburg, Ali Fröhlich, David Holland and Martin Taylor for their interest in and advice concerning Galois module structure. In addition, it was Martin Taylor who started me thinking about the Swan subgroup of a nilpotent group and about the problem of Terry Wall, concerning the Swan subgroup of a product, which appears in Chapter 5.

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