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1
Representations

Introduction

In this chapter we recall the most elementary facts about the finite-dimensional representations of a finite group over a field. In our applications we shall mainly be interested in representations which are realised over some subfield of the complex numbers.

Therefore, in Section 1, we introduce unitary, orthogonal and symplectic representations in preparation for our topological treatment of Brauer's Induction Theorem 2.1.20.

In Section 2 we specialise to the case of complex representations and recapitulate in brief the theory of the character (or trace) of a representation and the orthogonality relations which are satisfied by characters of irreducible representations. Induced representations are introduced and their standard adjointness properties (e.g. Frobenius reciprocity, the Double Coset formula) are derived.

Section 3 consists of exercises of an introductory nature.

1.1 Basic definitions

Let G be a finite group. Let K be a field and let V be a finite-dimensional vector space over K. Let $GL(V)$ denote the group of K-linear automorphisms of V. A homomorphism

1.1.1 $$\rho : G \longrightarrow GL(V)$$

gives rise to an action of G on V by means of K-linear automorphisms. Explicitly, if $g \in G$ and $v \in V$ the action is given by

1.1.2 $$g \cdot v = \rho(g)(v).$$

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A *finite-dimensional* K*-representation* of G is the K-isomorphism class of such an action. That is, the representation given by a G-action on V_1 is equivalent to the representation given by a G-action on V_2 if and only if there is a K-linear isomorphism, $\beta : V_1 \xrightarrow{\cong} V_2$, such that $\beta(g \cdot v_1) = g \cdot \beta(v_1)$ for all $g \in G$, $v_1 \in V_1$. In terms of homomorphisms, ρ, of 1.1.1 two homomorphisms

1.1.3 $$\rho_1, \rho_2 : G \longrightarrow GL(V)$$

give rise to the same representation if and only if there exists $B \in GL(V)$ such that $B\rho_1(g)B^{-1} = \rho_2(g)$ for all $g \in G$. Very often we will choose an isomorphism between V and K^n, where $dim_K(V) = n$ is the dimension of V. In that case a representation will become a conjugacy class of homomorphisms of the form

1.1.4 $$\rho : G \longrightarrow GL_n(K) = GL(K^n).$$

Example 1.1.5 (i) *One-dimensional representations* Let $K^* = K - 0 \cong GL_1(K)$ denote the multiplicative group of non-zero elements of K. A homomorphism, $\rho : G \longrightarrow K^*$, gives rise to a unique one-dimensional representation, since K^* is abelian.

(ii) *Permutation representations* Let G act on a finite set, X. Form the vector space, V, whose K-basis consists of the elements of X. Therefore G acts on V by permuting the basis vectors. In terms of a homomorphism into $GL_n(K)$, where $n = \#(X)$, we obtain a homomorphism, $\rho : G \longrightarrow GL_n(K)$, in which $\rho(g)$ has only one non-zero entry in each row or column. If we order the basis $\{x_i \in X ; 1 \leq i \leq n\}$ then the (i, j)th entry in $\rho(g)$ is 1 if $g(x_j) = x_i$ and zero otherwise. Such a matrix is called a *permutation matrix.*

1.1.6 If V is a G-representation then a K-subspace, W, of V which is preserved by the action of G is called a *subrepresentation.* For example, in the permutation representation of 1.1.5(ii) the subspace given by

$$W = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \sum \lambda_i = 0 \right\}$$

yields a subrepresentation.

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1.1.7 Functorial operations on vector spaces induce corresponding operations on G-representations. The direct sum, tensor product and the exterior power operations are three fundamental examples.

If V_1 and V_2 are two vector spaces with K-linear G-actions then so is the *direct sum*, $V_1 \oplus V_2$, if we define

$$g(v_1 \oplus v_2) = gv_1 \oplus gv_2 \quad (g \in G, v_i \in V_i).$$

In terms of matrices the direct sum of $\rho_i : G \longrightarrow GL_{n_i}(K)$ $(i = 1, 2)$ is given by the homomorphism

$$\rho_1 \oplus \rho_2 : G \longrightarrow GL_{n_1+n_2}(K)$$

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

The *tensor product*, $V_1 \otimes V_2$, is the vector space $Hom_K(Bil_K(V_1 \times V_2, K), K)$ in the finite-dimensional context. Here $Hom_K(V, W)$ denotes the space of K-linear maps from V to W and $Bil_K(V \times W, Z)$ denotes the space of K-bilinear maps. The dimension of $V_1 \otimes V_2$ is equal to $dim_K(V_1) \cdot dim_K(V_2)$, the product of the dimensions of V_1 and V_2.

Similarly we may define the tth *exterior power* of V, $\lambda^t(V)$, by setting

$$\lambda^t(V) = Hom_K(Alt_t(V), K),$$

where $Alt_K(V)$ denotes the subspace of multilinear maps,

$$f : V \times V \times \cdots \times V \longrightarrow K \ (t \text{ factors}),$$

such that

$$f(v_1, v_2, \ldots, v_t) = (-1)^{sign(\sigma)} f(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(t)}) \qquad (v_i \in V_i)$$

for any permutation, σ, of $\{1,\ldots,t\}$. When $char(K) \neq 2$, the dimension of $\lambda^t(V)$ is given by the binomial coefficient, $\binom{n}{t}$, where $dim_K(V) = n$.

In terms of matrix homomorphisms the matrix representing $V_1 \otimes V_2$ is the Kronecker product of the representing matrices, ρ_1 and ρ_2. When $t = dim_K(V)$ the matrix representing $\lambda^n(V)$ is given by $det(\rho) : G \longrightarrow K^{*}$ for ρ as in 1.1.4.

Theorem 1.1.8 (*Maschke's theorem*) *Suppose that the characteristic of K is prime to the order of G. Let V be a K-representation of G and let W be a subrepresentation. Then there exists a subrepresentation, W_1, such that $W \oplus W_1 = V$.*

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Proof There exists a K-linear map, $j : V \longrightarrow W$, such that $j(w) = w$ for all $w \in W$. If $j(g \cdot v) = g \cdot (j(v))$ for all $g \in G, v \in V$ then we could set $W_1 = ker(j)$, the kernel of j. However, j may not commute with the G-action so we replace it by $j_1 : V \longrightarrow W$, defined by $j_1(g) = (\#(G))^{-1}(\sum_{g\in G} g(j(g^{-1}(v))))$. Clearly $j_1(g \cdot v) = g \cdot (j_1(v))$ for all $g \in G, v \in V$ and if $w \in W$ then

$$j_1(w) = (\#(G))^{-1}\left(\sum_{g\in G} g(g^{-1}(v))\right) = w,$$

as required. □

1.1.9 A K-representation, V, of G is called *indecomposable* if an isomorphism of the form $V = W \oplus W_1$, for subrepresentations W and W_1, implies either that $W = 0$ or $W = V$. V is called *irreducible* if V has no subrepresentations except $\{0\}$ and V. Theorem 1.1.8 states that these notions coincide when the order of G is prime to the characteristic of K, which is the situation which will mainly concern us.

1.1.10 We will often be dealing with representations afforded by vector spaces over the field, $\mathbf{C}$, of complex numbers. The group, $GL_n(\mathbf{C})$ has a number of compact subgroups which are of special interest. The *unitary group*, $U(n)$, is defined to be

1.1.11 $$U(n) = \{X \in GL_n(\mathbf{C}) | XX^* = I_n\},$$

where I_n is the $n \times n$ identity matrix and X^* is the matrix whose (i, j)th entry is $\overline{X}_{ji}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$. A *unitary representation* will mean the $U(n)$-conjugacy class of a homomorphism of the form

1.1.12 $$\rho : G \longrightarrow U(n).$$

Clearly, each unitary representation gives rise to a $\mathbf{C}$-representation of G and this induces a one-one correspondence between $U(n)$-representations and n-dimensional $\mathbf{C}$-representations (see 1.3.1). Note that $U(n)$ is the subgroup of $GL_n(\mathbf{C})$ consisting of matrices which preserve the semi-linear inner product on $\mathbf{C}^n$ given by $< \underline{x}, \underline{y} >= \sum_{i=1}^n x_i \cdot \overline{y}_i$, where $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$.

Similarly, if $\mathbf{R}$ is the field of real numbers then the *orthogonal group*, $O(n)$, is the subgroup of matrices which preserve the inner product $< \underline{x}, \underline{y} >= \sum_{i=1}^n x_i \cdot y_i$ so that

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1.1.13 $$O(n) = GL_n(\mathbf{R}) \bigcap U(n).$$

An *orthogonal representation* is an $O(n)$-conjugacy class of a homomorphism of the form

1.1.14 $$\rho : G \longrightarrow O(n).$$

As in the complex case there is a one–one correspondence between $O(n)$-representations and n-dimensional **R**-representations of G (see 1.3.2).

Let **H** denote the quaternion skew-field. If $z = a + ib + jc + kd$ is a quaternion then $\overline{z} = a - ib - jc - kd$. On $\mathbf{H}^n$ we have an inner product given by $< \underline{x}, \underline{y} >= \sum_{i=1}^n x_i \cdot \overline{y}_i$ for $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$. The group of (left) **H**-semilinear automorphisms of $\mathbf{H}^n$ which preserve this form is called the *symplectic group*, $Sp(n)$. As a **C**-vector space **H** is isomorphic to $\mathbf{C}^2$ so that $Sp(n)$ is a subgroup of $U(2n)$. A *symplectic representation* is a conjugacy class of homomorphisms of the form

1.1.15 $$\rho : G \longrightarrow Sp(n).$$

1.2 Complex representations

In this section we shall restrict our attention to **C**-representations of G. Given a representation of the form of 1.1.4

1.2.1 $$\rho : G \longrightarrow GL_n(\mathbf{C}),$$

we define the *character* of ρ, χ_ρ, to be the **C**-valued function given by

1.2.2 $$\chi_\rho(g) = Trace(\rho(g)) = \textstyle\sum_{i=1}^n \rho(g)_{ii} \quad (g \in G).$$

Proposition 1.2.3 (i) *χ_ρ depends only on the class of ρ in 1.2.1 as a representation.*

(ii) *$\chi_\rho(g)$ is the sum of the eigenvalues of $\rho(g)$, counted with their multiplicities.*

(iii) *$\chi_\rho(1) = dim(\rho) = n$.*

(iv) *$\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.*

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Proof If $X \in GL_n(\mathbf{C})$ set $\phi = X\rho X^{-1}$ then we must verify that $\chi_\phi = \chi_\rho$. However, if t is an indeterminate,

$$\begin{aligned} det(tI_n - \phi(g)) &= det(X(tI_n - \rho(g))X^{-1}) \\ &= det(tI_n - \rho(g)) \\ &= t^n - Trace(\rho(g))t^{n-1} + \ldots, \end{aligned}$$

which proves part (i).

Part (ii) follows by conjugating $\rho(g)$ into its Jordan canonical form, which is an upper triangular matrix whose diagonal entries are the eigenvalues. Each eigenvalue, λ, appears on the diagonal m_λ times, where m_λ is the multiplicity of λ.

Part (iii) is clear, since $Trace(I_n) = n$ and part (iv) follows from part (ii), since the eigenvalues of $\rho(g^{-1}) = \rho(g)^{-1}$ are the inverses of those for $\rho(g)$. However, since $\rho(g)$ has finite order, these eigenvalues are complex numbers of unit norm and therefore the inverse of each eigenvalue is equal to its complex conjugate. □

Proposition 1.2.4 *Let $\rho_i : G \longrightarrow GL(V_i)$ $(i = 1, 2)$ be two representations with character functions, $\chi_i = \chi_{\rho_i}$. Then*

(i) *$\chi_1 + \chi_2$ is the character of $V_1 \oplus V_2$ and*

(ii) *$(\chi_1) \cdot (\chi_2)$ is the character of $V_1 \otimes V_2$.*

Proof For $g \in G$ let $\rho_1(g) = X$ and $\rho_2(g) = Y$ then $(\rho_1 \oplus \rho_2)(g)$ is given by the matrix

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

whose trace is clearly equal to $Trace(X) + Trace(Y)$.

The tensor product, $\rho_1(g) \otimes \rho_2(g)$, is given by the Kronecker product of X and Y, by 1.3.3. This is the matrix of the form

$$\begin{pmatrix} X \cdot y_{11} & X \cdot y_{12} & \ldots & X \cdot y_{1n} \\ X \cdot y_{21} & X \cdot y_{22} & \ldots & X \cdot y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ X \cdot y_{n1} & X \cdot y_{n2} & \ldots & X \cdot y_{nn} \end{pmatrix},$$

where $dim_{\mathbf{C}}(\rho_2) = n$. The trace of this matrix is clearly equal to

$$Trace(X) \cdot [y_{11} + y_{22} + \ldots + y_{nn}].$$

□

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Lemma 1.2.5 (*Schur's Lemma*) *Let V_1 and V_2 be irreducible representations of G. Let $Hom_G(V_1, V_2) =$*

$$\{f : V_1 \longrightarrow V_2 \text{ linear} \mid f(gv_1) = gf(v_1);\ \ g \in G, v_1 \in V_1\}.$$

Then

(i) *if $V_1 \neq V_2$, then $Hom_G(V_1, V_2) = 0$,*
(ii) *if $V_1 = V_2$,*

$$Hom_G(V_1, V_1) = \{f \mid f(x) = \lambda \cdot x \text{ for some } \lambda \in \mathbf{C}\}.$$

Proof If $f : V_1 \longrightarrow V_2$ is non-zero then $ker(f)$ and $im(f)$ are subrepresentations of V_1 and V_2 respectively. Since $f \neq 0$, $ker(f) \neq V_1$ and $im(f) \neq 0$ so that $ker(f) = 0$ and $im(f) = V_2$, which means that f is an isomorphism. This means that $V_1 = V_2$ as representations. However, when $V_1 = V_2$, let λ be an eigenvalue for f. The subspace

$$W = \{v_1 \in V_1 \mid f(v_1) = \lambda \cdot v_1\}$$

is a non-zero subrepresentation of V_1 so that $V_1 = W$, as required. □

Corollary 1.2.6 *Let V_1,V_2 be as in 1.2.5. Let $f : V_1 \longrightarrow V_2$ be a linear map. Define $F : V_1 \longrightarrow V_2$ by*

$$F(v_1) = \#(G)^{-1}\left(\sum_{g\in G} g \cdot f(g^{-1} \cdot v_1)\right).$$

Then

(i) $F = 0$ *if $V_1 \neq V_2$ and*
(ii) $F(v_1) = (dimV_1)^{-1} Trace(f) \cdot v_1$ *if $V_1 = V_2$.*

Proof Clearly $F(g \cdot v_1) = g \cdot F(v_1)$ for all $g \in G$ so that $F = 0$ unless $V_1 = V_2$. If $V_1 = V_2$ then $F(v_1) = \lambda \cdot v_1$ for all $v_1 \in V_1$ and, by taking traces,

$$(dimV_1) \cdot \lambda = \#(G)^{-1}\left(\sum_{g\in G} Trace(g \cdot f(g^{-1} \cdot -))\right) = Trace(f)$$

since $Trace(g \cdot f(g^{-1} \cdot -)) = Trace(f)$ for all $g \in G$. □

Definition 1.2.7 Suppose that W_1, W_2 are two representations with characters χ_1, χ_2 respectively. The *Schur inner product* $< W_1, W_2 > =$

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$< \chi_1, \chi_2 >$ is defined by

$$< \chi_1, \chi_2 >= \#(G)^{-1} \left(\sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} \right).$$

Theorem 1.2.8 *In* 1.2.7

(i) $< \chi_1, \chi_2 >= dim_{\mathbf{C}} Hom_G(W_1, W_2)$,

(ii) $< \chi_1, \chi_2 >=< \chi_2, \chi_1 >$,

(iii) $< \chi_1, \chi_1 >= 1$ *if and only if* W_1 *is irreducible.*

Proof Suppose that $A_1, \ldots, A_s, B_1, \ldots, B_t$ are irreducible representations such that

$$W_1 = \oplus_{i=1}^s A_i, W_2 = \oplus_{j=1}^t B_j,$$

then $Hom_G(W_1, W_2) \cong \oplus_{i,j} Hom_G(A_i, B_j)$. Now let χ_{A_i}, χ_{B_j} denote the characters of these irreducible representations. By 1.2.4, $\chi_1 = \sum_{i=1}^s \chi_{A_i}$ and $\chi_2 = \sum_{j=1}^t \chi_{B_j}$ so that

$$< \chi_1, \chi_2 >= \sum_{i,j} < \chi_{A_i}, \chi_{B_j} > .$$

Therefore, in order to prove part (i), we may assume that W_1 and W_2 are irreducible.

Choose bases for W_1 and W_2 so that the representation, W_i, corresponds to $\rho_i : G \longrightarrow GL_{n_i}(\mathbf{C})$. In Corollary 1.2.6 assume that f is represented by a matrix, X. In this notation 1.2.6 becomes

1.2.9 $$\#(G)^{-1} \textstyle\sum_{a,b} \sum_{g \in G} \rho_2(g)_{ia} X_{ab} \rho_1(g)^{-1}_{bj}$$

$$= \begin{cases} 0 & \text{if } i \neq j \text{ or } W_1 \neq W_2 \\ (dim(W_1))^{-1} \cdot Trace(X) & \text{if } W_1 = W_2 \text{ and } i = j. \end{cases}$$

When $W_1 \neq W_2$ choose f with $X_{ab} = 0$ except for $X_{ij} = 1$ so that 1.2.9 implies that

$$\sum_{g \in G} \rho_2(g)_{ii} \rho_1(g^{-1})_{jj} = 0 \text{ for all } i, j.$$

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Therefore

$$\begin{aligned} \#(G) < \chi_1, \chi_2 > &= \textstyle\sum_{g\in G} \chi_1(g) \cdot \overline{\chi_2(g)} \\ &= \textstyle\sum_{g\in G} \sum_i \sum_j \rho_1(g)_{ii} \cdot \rho_2(g^{-1})_{jj}, \quad \text{by 1.2.3(iv)}, \\ &= \textstyle\sum_{ij} \sum_{g\in G} \rho_1(g^{-1})_{ii} \rho_2(g)_{jj} \\ &= 0 \\ &= dim_{\mathbf{C}} Hom_G(W_1, W_2). \end{aligned}$$

When $W_1 = W_2$ we may again take $X_{ab} = 0$ except for $X_{ij} = 1$ to obtain

$$\sum_{g\in G} \rho_2(g)_{ii} \cdot \rho_1(g_{jj}^{-1}) = \begin{cases} 0 & \text{if } i \neq j, \\ \#(G) \cdot (dim W_1)^{-1} & \text{if } i = j. \end{cases}$$

Therefore

$$\begin{aligned} \#(G) \cdot < \chi_1, \chi_2 > &= \textstyle\sum_{i,j} \sum_{g\in G} \rho_1(g^{-1})_{ii} \cdot \rho_2(g)_{jj} \\ &= \#(G) \end{aligned}$$

and

$$1 =< \chi_1, \chi_2 >=< \chi_1, \chi_1 >,$$

as required.

To prove part (ii) we remark that $< \chi_2, \chi_1 >$ is the complex conjugate of $< \chi_1, \chi_2 >$, by definition, but $< \chi_1, \chi_2 >$ is a positive integer and therefore is fixed by complex conjugation.

Part (iii) follows from part (i), 1.2.5 and the observation that

$$dim_{\mathbf{C}} Hom_G(W_1, W_1) \geq \sum_{i=1}^{s} Hom_G(A_i, A_i) = s.$$

□

Corollary 1.2.10 *In the notation of* 1.2.7, $W_1 = W_2$ *as representations if and only if the characters* χ_1 *and* χ_2 *are equal.*

Proof Firstly, observe that

$$dim_{\mathbf{C}} W_1 = \chi_1(1) = \chi_2(1) = dim_{\mathbf{C}} W_2,$$

so that we may prove this by induction on dimension. When $dim_{\mathbf{C}} W_1 = 1$ both representations are irreducible and

$$dim_{\mathbf{C}} Hom_G(W_1, W_2) = dim_{\mathbf{C}} Hom_G(W_1, W_1) = 1,$$

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by 1.2.8(i) so that, by 1.2.5, there is a G-isomorphism between W_1 and W_2. Now let A_1 be an irreducible summand of W_1 and set

$$W_2 = \oplus_{j=1}^{t} B_j,$$

as in the proof of 1.2.8. Hence

$$dim_{\mathbf{C}} Hom_G(A_1, W_2) = dim_{\mathbf{C}} Hom_G(A_1, W_1) \geq 1,$$

by 1.2.8(i). Therefore, for some j, $Hom_G(A_1, B_j) \neq 0$. By 1.1.8, $W_2 \cong A_1 \oplus W_2'$ and $W_1 \cong A_1 \oplus W_1'$. The characters of W_1' and W_2' are therefore equal, by 1.2.4, so that $W_1' \cong W_2'$ as representations, which completes the proof. □

Corollary 1.2.11 *If W and V are representations of G such that V is irreducible then $< W, V >$ is equal to the* multiplicity of V in W (*i.e. the number of times that V appears in a decomposition of W as a sum of irreducibles*).

Proof If $W = nV \oplus V_1 \oplus \ldots \oplus V_t$ with V_i irreducible and not equivalent to V then

$$\begin{aligned} < W, V > &= < V, W > \\ &= dim_{\mathbf{C}}(Hom_G(V, W) \\ &= n dim_{\mathbf{C}}(Hom_G(V, V) + \textstyle\sum_{i=1}^{t} dim_{\mathbf{C}}(Hom_G(V, V_i)) \\ &= n, \qquad \text{by 1.2.5} \end{aligned}$$

□

Definition 1.2.12 The *regular representation* of G is the permutation representation obtained, by the method of 1.1.5(ii), from the set G together with the G-action given by left multiplication. The regular representation will be denoted by $Ind_{\{1\}}^{G}(1)$ in recognition of its construction as an induced representation (see 1.2.31).

Proposition 1.2.13 *Let r_G denote the character of the regular representation. Then*

$$r_G(g) = \begin{cases} 0 & if \;\; g \neq 1, \\ \#(G) & if \;\; g = 1. \end{cases}$$

Proof By definition (see 1.1.4(ii)) a basis for $Ind_{\{1\}}^{G}(1)$ consists of $\{x \mid x \in G\}$. If $g \neq 1$ then $g \cdot x \neq x$ so that the trace of multiplication by g is

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