

## Contents

<i>Frontispiece: let <math>A</math> be a ring and <math>M</math> an <math>A</math>-module ...</i>	<i>page</i> iv
<i>Illustrations</i>	xi
<i>Preface</i>	xiii
<b>0 Hello!</b>	1
0.1 Where we're going	1
0.2 Some definitions	2
0.3 The elementary theory of factorisation	2
0.4 A first view of the bridge	3
0.5 The geometric side – the case of a hypersurface	3
0.6 $\mathbb{Z}$ versus $k[X]$	5
0.7 Examples	7
0.8 Reasons for studying commutative algebra	10
0.9 Discussion of contents	12
0.10 Who the book is for	13
0.11 What you're supposed to know	13
Exercises to Chapter 0	14
<b>1 Basics</b>	19
1.1 Convention	19
1.2 Ideals	19
1.3 Prime and maximal ideals, the definition of $\text{Spec } A$	20
1.4 Easy examples	21
1.5 Worked examples: $\text{Spec } k[X, Y]$ and $\text{Spec } \mathbb{Z}[X]$	22
1.6 The geometric interpretation	23
1.7 Zorn's lemma	25
1.8 Existence of maximal ideals	26

viii	<i>Contents</i>	
1.9	Plenty of prime ideals	27
1.10	Nilpotents and the nilradical	27
1.11	Discussion of zerodivisors	28
1.12	Radical of an ideal	29
1.13	Local ring	31
1.14	First examples of local rings	31
1.15	Power series rings and local rings	32
	Exercises to Chapter 1	33
<b>2</b>	<b>Modules</b>	<b>37</b>
2.1	Definition of a module	37
2.2	Harmless formalism	37
2.3	The homomorphism and isomorphism theorems	38
2.4	Generators of a module	40
2.5	Examples	41
2.6	The Cayley–Hamilton theorem	41
2.7	The determinant trick	43
2.8	Corollaries – Nakayama’s lemma	43
2.9	Exact sequences	44
2.10	Split exact sequences	45
	Exercises to Chapter 2	46
<b>3</b>	<b>Noetherian rings</b>	<b>49</b>
3.1	The ascending chain condition	49
3.2	Noetherian rings	50
3.3	Examples	51
3.4	Noetherian modules	52
3.5	Properties of Noetherian modules	53
3.6	The Hilbert basis theorem	54
	Exercises to Chapter 3	55
<b>4</b>	<b>Finite extensions and Noether normalisation</b>	<b>58</b>
4.1	Finite and integral $A$ -algebras	59
4.2	Finite versus integral	60
4.3	Tower laws	61
4.4	Integral closure	61
4.5	Preview: nonsingularity and normal rings	62
4.6	Noether normalisation	63
4.7	Proof of Claim	64
4.8	Another proof of Noether normalisation	65
4.9	Field extensions	66

<i>Contents</i>	ix
4.10 The weak Nullstellensatz	67
Exercises to Chapter 4	67
<b>5 The Nullstellensatz and Spec <math>A</math></b>	<b>70</b>
5.1 Weak Nullstellensatz	70
5.2 Maximal ideals of $k[X_1, \dots, X_n]$ and points of $k^n$	70
5.3 Definition of a variety	71
5.4 Remark on algebraically nonclosed $k$	72
5.5 The correspondences $V$ and $I$	72
5.6 The Nullstellensatz	73
5.7 Irreducible varieties	74
5.8 The Nullstellensatz and Spec $A$	75
5.9 The Zariski topology on a variety	75
5.10 The Zariski topology on a variety is Noetherian	76
5.11 Decomposition into irreducibles	76
5.12 The Zariski topology on a general Spec $A$	77
5.13 Spec $A$ for a Noetherian ring	78
5.14 Varieties versus Spec $A$	80
Exercises to Chapter 5	82
<b>6 Rings of fractions <math>S^{-1}A</math> and localisation</b>	<b>84</b>
6.1 The construction of $S^{-1}A$	84
6.2 Easy properties	86
6.3 Ideals in $A$ and $S^{-1}A$	87
6.4 Localisation	88
6.5 Modules of fractions	89
6.6 Exactness of $S^{-1}$	90
6.7 Localisation commutes with taking quotients	91
6.8 Localise and localise again	92
Exercises to Chapter 6	92
<b>7 Primary decomposition</b>	<b>95</b>
7.1 The support of a module $\text{Supp } M$	96
7.2 Discussion	97
7.3 Definition of $\text{Ass } M$	98
7.4 Properties of $\text{Ass } M$	99
7.5 Relation between $\text{Supp}$ and $\text{Ass}$	100
7.6 Disassembling a module	103
7.7 The definition of primary ideal	103
7.8 Primary ideals and $\text{Ass}$	105
7.9 Primary decomposition	105

x	<i>0 Contents</i>	
7.10	Discussion: motivation and examples	106
7.11	Existence of primary decomposition	108
7.12	Primary decomposition and $\text{Ass}(A/I)$	109
7.13	Primary ideals and localisation	109
	Exercises to Chapter 7	110
<b>8</b>	<b>DVRs and normal integral domains</b>	<b>112</b>
8.1	Introduction	112
8.2	Definition of DVR	113
8.3	A first criterion	113
8.4	The Main Theorem on DVRs	114
8.5	General valuation rings	116
8.6	Examples of general valuation rings	117
8.7	Normal is a local condition	118
8.8	A normal ring is a DVR in codimension 1	119
8.9	Geometric picture	121
8.10	Intersection of DVRs	121
8.11	Finiteness of normalisation	122
8.12	Proof of Theorem 8.11	123
8.13	Appendix: Trace and separability	124
	Exercises to Chapter 8	126
<b>9</b>	<b>Goodbye!</b>	<b>129</b>
9.1	Where we've come from	129
9.2	Where to go from here	130
9.3	Tidying up some loose ends	132
9.4	Noetherian is not enough	135
9.5	Akizuki's example	139
9.6	Scheme theory	141
9.7	Abstract versus applied algebra	142
9.8	Sketch history	143
9.9	The problem of algebra in teaching	144
9.10	How the book came to be written	145
	Exercises to Chapter 9	146
	<i>Bibliography</i>	149
	<i>Index</i>	150