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0521458897 - Undergraduate Commutative Algebra
Miles Reid
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Commutative algebra is at the crossroads of algebra, number theory and algebraic geometry. This textbook, intended for advanced undergraduate or beginning graduate students with some previous experience of rings and fields, covers roughly the same material as Chapters 1–8 of Atiyah and Macdonald [A & M], but is cheaper, has more pictures, and is considerably more opinionated.

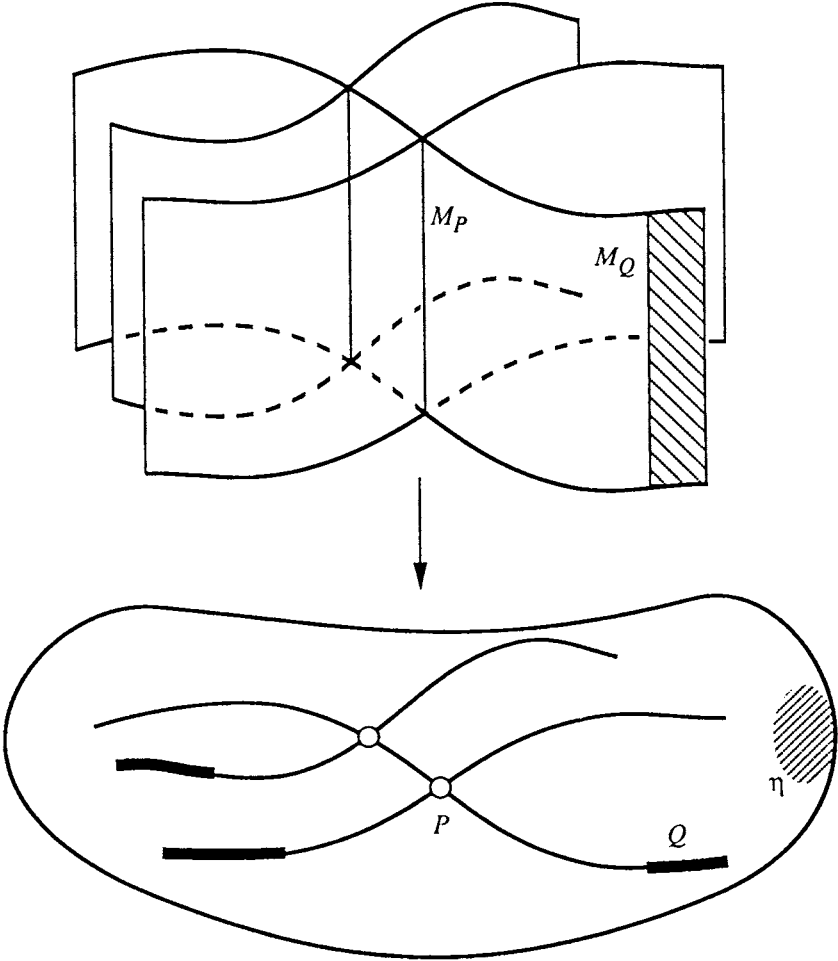
Alongside standard algebraic notions such as generators of modules and the ascending chain condition, the book develops in detail the geometric view of a commutative ring as the ring of functions on a space. The starting point is the Nullstellensatz, which provides a close link between the geometry of a variety V and the algebra of its coordinate ring $A = k[V]$; however, many of the geometric ideas arising from varieties apply also to fairly general rings.

The final chapter relates the material of the book to more advanced topics in commutative algebra and algebraic geometry. It includes an account of some famous “pathological” examples of Akizuki and Nagata, and a brief but thought-provoking essay on the changing position of abstract algebra in today’s world.

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Preface

These are notes from a commutative algebra course taught at the University of Warwick several times since 1978. In addition to standard material, the book contrasts the methods and ideology of abstract algebra as practiced in the 20th century with its concrete applications in algebraic geometry and algebraic number theory.