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Contents

Introduction

Invariant theory is a subject with a long history, a subject which in the words of Dieudonné and Carrell [32] "has already been pronounced dead several times, and like the Phoenix it has been again and again arising from the ashes."

The starting point is a linear representation of a linear algebraic group on a vector space, which then induces an action on the ring of polynomial functions on the vector space, and one looks at the ring consisting of those polynomials which are invariant under the group action. The reason for restricting ones attention to linear algebraic group representations, or equivalently to Zariski closed subgroups of the general linear group on the vector space, is that the polynomial invariants of an arbitrary subgroup of the general linear group are the same as the invariants of its Zariski closure.

In the nineteenth century, attention focused on proving finite generation of the algebra of invariants by finding generators for the invariants in a number of concrete examples. One of the high points was the proof of finite generation for the invariants of $SL_2(\mathbb{C})$ acting on a symmetric power of the natural representation, by Gordan [39] (1868). The subject generated a language all its own, partly because of the influence of Sylvester, who was fond of inventing words to describe rather specialized concepts.

In the late nineteenth and early twentieth century, the work of David Hilbert and Emmy Noether in Göttingen clarified the subject considerably with the introduction of abstract algebraic machinery for addressing questions like finite generation, syzygies, and so on. This was the beginning of modern commutative algebra. Hilbert proved finite generation for the case of $GL_n(\mathbb{C})$ acting on a symmetric power of the natural representation [45] (1890), using the fact that a polynomial ring is Noetherian (Hilbert's basis theorem).

Hilbert's fourteenth problem, posed at the 1900 International Congress of Mathematicians in Paris, asks whether the invariants are always finitely generated. Well, actually, this is false. He was under the impression that L. Maurer had just proved this, and he goes on to ask a more complicated question which generalizes this. In fact, Maurer's proof contained a mistake, and Nagata [65] (1959) found a counterexample to finite generation. An account of the Hilbert problems and the progress made on them up to the mid nineteen seventies is contained in the A.M.S. publication [6].

For reductive algebraic groups, however, finite generation does hold. This was first proved over the complex numbers by Weyl [110] (1926), using his "unitarian trick", which amounts to proving that reductive groups are linearly reductive. In characteristic p, reductive groups are no longer necessariy linearly reductive (the only connected linearly reductive groups in characteristic p are the tori), but Mumford [64] conjectured and Haboush [41] (1975) proved that reductive groups are "geometrically reductive", and Nagata [67] had already proved that this is enough to ensure finite generation for the invariants.

The question of when the ring of invariants of representation of a finite group over the complex numbers is a polynomial ring was solved by Shephard and Todd [95] (1954). Their theorem states that the ring of invariants is a polynomial ring if and only if the group is generated by elements fixing a hyperplane. Such elements are called pseudoreflections. Their proof involved classifying the finite groups generated by pseudoreflections, and is a tour de force of combinatorics and geometry. The classification was used to prove that groups generated by pseudoreflections have invariant rings that are polynomial rings, and this first implication together with Molien's theorem was then used in the proof of the converse. Later, Chevalley [26] (1955) provided a proof of the first implication for a real reflection group, which avoided classifying the groups. This proof involved the combinatorics of differential operators. Serre later showed how this proof could be adapted to the complex case. A more homological proof of this first implication was found by Larry Smith [96] (1985), and a proof of the converse which works in arbitrary characteristic appeared in Bourbaki [17] Chapter 5 §5, Exercise 7 (1981). The first implication is false in arbitrary characteristic.

The next interesting ring theoretic property of the ring of invariants of a reductive group is that it is Cohen–Macaulay. This was proved in the case of a finite group by Hochster and Eagon [47] (1971). Watanabe [107, 108] (1974) proved that in this case the ring of invariants is Gorenstein provided the group acts as matrices with determinant one. In the general reductive case, both these statements were proved by Hochster and Roberts [48] (1974).

The question of when unique factorization holds in rings of invariants was first attacked by Samuel [86] (1964). He developed the framework for defining the ideal class group of a Krull domain. A finitely generated Krull domain is the same thing as a normal domain (i.e., a Noetherian integrally closed domain). Using Samuel's ideas, Nakajima proved that the ring of invariants of a finite group is a unique factorization domain if and only if there are no non-trivial homomorphisms from the group to the multiplicative group of the field, taking every pseudoreflection to the identity element.

A lot of the ideas involved in invariant theory in the reductive case are already present in the finite case; and the latter has the advantage that one does not have to spend an inordinate amount of time setting up the machinery of algebraic groups before getting to the interesting theorems. For this reason, I decided to restrict the scope of this book to the case of finite group representations.

This book is based on a lecture course I gave at Oxford in the spring of 1991. My starting point for these lectures was the excellent survey article of Richard Stanley [102], which I strongly recommend to anyone wishing to get an overview of the subject. The influence of this article will be apparent in almost every part of this book. The main direction in which this book differs from that article, apart from length and amount of detail, is that I have tried to indicate how much is true in arbitrary characteristic, whereas Stanley restricts his discussion to characteristic zero. Also, Stanley does not discuss divisor classes and unique factorization.

At Larry Smith's invitation, I spent two months in Göttingen in the summer of 1991, and he persuaded me that we should write a book together on invariant theory of finite groups. Differences in personal style prevented this from being realized as a joint project, but I would like to take this opportunity to thank him for discussions which had a considerable effect on the presentation of the material in Sections 7.2 and 8.1, and indeed for persuading me to embark on this project in the first place.

I would also like to thank Bill Crawley–Boevey for taking the trouble to make many constructive criticisms, and for suggesting the inclusion of the appendices; and David Tranah of Cambridge University Press for his usual endless supply of patience.