1

Modelling a viscous fluid

1.1 Modelling aspects

The principal motivation for the study of viscous fluid dynamics is the inability of the Euler equations of inviscid flow to predict certain familiar phenomena. We begin by reviewing these equations and some of their successes and failures.

In the simplest case of incompressible flow of an inviscid fluid, the Euler equations are

$$\left.\begin{aligned} \nabla.\mathbf{u} &= 0 \\ \text{and} \qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u}\right) &= -\nabla p + \mathbf{F} \end{aligned}\right\} \qquad (1.1)$$

for the velocity $\mathbf{u}$ and pressure p. Here the density ρ is constant and we assume that the body force $\mathbf{F}$ is prescribed. If $\mathbf{F}$ is conservative then these equations lead to the remarkable result that any flow that is initially irrotational will remain irrotational. This means that, for a wide range of flows of practical interest, $\mathbf{u} = \nabla\phi$ where ϕ is determined by the equation

$$\nabla^2\phi = 0 \qquad (1.2)$$

together with suitable boundary conditions. This enables an enormous number of predictions to be made about inviscid flow which extend far beyond the simple examples found in textbooks on hydrodynamics. Some of the notable successes include free surface flows [Stoker], separated and bubbly flows [Birkhoff and Zarantonello] and aerodynamic flows (away from 'boundary layers' and 'wakes') [Milne-Thomson]. When compressibility is taken into account, by allowing ρ to vary and adding an equation of state to the model, even more phenomena can be described accurately. In particular the whole subject of acoustics is modelled in this way.

In spite of these successes there are many familiar situations which cannot be explained by equations (1.1) as illustrated by the following examples.

1. The first example comes from the classical theory of flight. It is possible to derive a theory of flow past aerofoils based on incompressible flow and using equations (1.1) but, unless we introduce an extra modelling assumption (the Kutta-Joukowski hypothesis), we encounter the D'Alembert paradox which precludes both lift and drag on either an obstacle placed in a uniform stream or an object travelling with constant velocity. The Kutta-Joukowski hypothesis, by introducing circulation around the object, does overcome this limitation but the origin of this circulation cannot be explained from the inviscid flow equations alone.

2. An everyday example is the observation that dust cannot be cleaned off the smooth surfaces of a car by simply driving fast. Inviscid theory for the airflow predicts a nonzero tangential component of velocity at a surface which the existence of the dust layer belies.

3. A common practical use for fluids is in lubricated bearings. The ability of a thin layer of fluid to support a large normal load while offering very little resistance to tangential motion is crucial in many kinds of machinery but cannot be explained by equations (1.1).

4. Another phenomenon observed in fluids is that they can be heated if external work is done on them. For compressible fluids, like the air in a bicycle pump, this can be explained by inviscid theory but for an incompressible inviscid fluid with boundaries held at constant temperature the model predicts that the temperature will remain constant throughout. This is in contradiction to experience in many situations; for example the oil in a bearing or fluid which is being injected under pressure into a thin mould can become significantly hotter without any change in density.

5. Another example comes from the theory of flight in the upper atmosphere. At altitudes between 10 and 100 km, the mean free path of a molecule in the atmosphere is so great that the macroscopic continuum model (1.1) is invalid. In this situation, as in some very small scale flows such as those involving free Brownian motion, some specific consideration must be given to the particle motion. This difficulty can only be dealt with via statistical mechanics and is beyond the scope of this book. However the idea of viscosity is relevant here too, as will be mentioned later in this chapter.

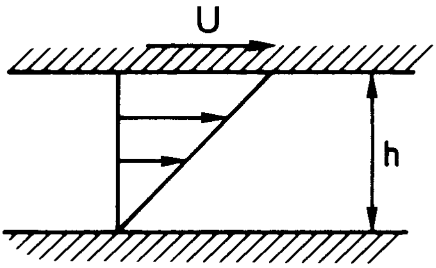

Fig. 1.1. Shearing flow

All these examples indicate the need for another model. The first four examples are all cases where *viscosity* or internal friction needs to be taken into account, and the aim of this book is to derive a model that will explain these phenomena. To do this we need to make just two basic experimental observations about how a viscous fluid reacts to shear forces and normal forces.

1.1.1 Shear forces

It is straightforward to set up an experiment to show that many fluids resist shear. In figure 1.1 the top plate is moved with speed U over a layer of viscous fluid of depth h lying on a fixed plate. The force required is found to be proportional to U/h, whereas for an inviscid fluid satisfying equations (1.1), the force required would be zero. A *viscometer* is a device which measures the viscosity of a fluid and is frequently based on this experiment. The constant of proportionality observed above is a direct measure of the viscosity of the fluid.

1.1.2 Normal forces

It is also easy to see that viscous fluids resist normal loading. In figure 1.2 the fluid (perhaps toffee) is pulled apart with velocity U and it is found that the required force is proportional to U/L.

These two simple observations will enable us to derive a model for viscous flow and this modelling is one of the most crucial aspects of the subject. The rest of this chapter will be devoted to modelling viscous flow, but at the end of this book (Chapter 5) we will consider briefly some other models of continua which can be thought of as generalisations of inviscid flows.

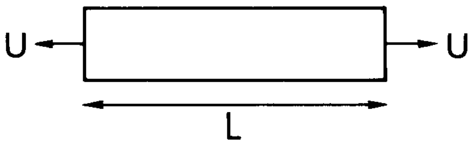

Fig. 1.2. Extensional flow

1.2 Stress

The first piece of evidence from the viscometry experiments in §1.1 is that the direction of the force exerted by the fluid on one side of a small surface element on the fluid on the other side of the element is not known a priori as it is for an inviscid flow. Therefore we need to define quantities σ_{ij} which are the force per unit area in the x_i direction[1] acting on a surface element whose normal is in the x_j direction. For an inviscid fluid, $\sigma_{ij} = -p\delta_{ij}$ where p is the pressure and δ_{ij} is the Kronecker delta defined by $\delta_{ij} = \{1 \text{ if } i = j,\ 0 \text{ if } i \neq j\}$. At first sight the introduction of σ_{ij} in place of the single variable p makes the modelling of a viscous flow a formidable task. However, we will see that we can work with just the nine quantities σ_{ij} referred to *one* specific set of axes and that, after certain assumptions, they can be written very simply in terms of derivatives of the velocity components. First, we need to show that, the 'stress' (i.e. force per unit area) on an *arbitrarily oriented* surface can be found in terms of these nine quantities by the application of Newton's law of motion to the tetrahedron of fluid shown in figure 1.3. If the force per unit area on triangle ABC is $\mathbf{F} = (\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3)$ then

$$F_i \Delta ABC - \sigma_{i1}\Delta OBC - \sigma_{i2}\Delta OAC - \sigma_{i3}\Delta OAB =$$
$$\rho \,\mathrm{vol}\, OABC \times \text{ acceleration in } x_i \text{ direction.}$$

If we now shrink the tetrahedron to zero and assume that the acceleration remains finite we obtain

$$F_i dS = \sigma_{i1} dS_1 + \sigma_{i2} dS_2 + \sigma_{i3} dS_3$$

where dS_1, dS_2, dS_3 are the areas of triangles OBC, OAC, OAB, respectively and dS is the area of triangle ABC. If $\mathbf{n}$ is the unit outward normal to ΔABC, $dS_i = n_i dS$ and so

$$F_i = \sigma_{i1}n_1 + \sigma_{i2}n_2 + \sigma_{i3}n_3 = \sigma_{ij}n_j \tag{1.3}$$

[1] We will try to use the notation (x, y, z) for Cartesian coordinates throughout these notes but in this chapter suffices and the summation convention are unavoidable for reasons of space; hence $x = x_1$, $y = x_2$, $z = x_3$.

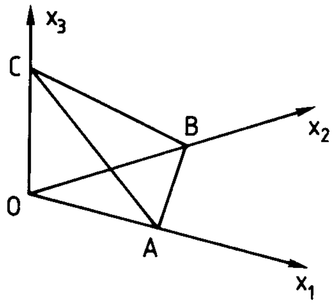

Fig. 1.3. Forces on a fluid tetrahedron

on using the summation convention. Thus we have found the stress on *any* surface element at *any* point in terms of the nine quantities σ_{ij}. These quantities form a 3×3 array, S, which, by using (1.3), can also be thought of as a matrix or linear transformation as follows.

We write (1.3) as $\mathbf{F} = S\mathbf{n}$ and let $T = \{t_{ij}\}$ be an orthogonal transformation which takes $\mathbf{x}$ into $\mathbf{x}' = T\mathbf{x}$. Then $\mathbf{F}$ and $\mathbf{n}$ will transform into $\mathbf{F}'$ and $\mathbf{n}'$ and (1.3) will be replaced by $\mathbf{F}' = S'\mathbf{n}'$. Substituting for these dashed variables in terms of the original ones and using (1.3) leads to

$$TS\mathbf{n} = S'T\mathbf{n}$$

and hence, since $\mathbf{n}$ is an arbitrary unit vector,

$$S' = TST^T \qquad (1.4)$$

or, equivalently,

$$\sigma'_{ij} = t_{i\alpha}t_{j\beta}\sigma_{\alpha\beta}. \qquad (1.5)$$

Thus σ_{ij} satisfies the usual rule for a linear transformation or matrix in a vector space (i = row, j = column). This is also the definition of a second rank *tensor*[2] and σ_{ij} is known as the *stress tensor*.

We can further simplify our modelling task by noting that we can reduce the nine quantities in σ_{ij} to six by consideration of the angular momentum of a fluid element. A general account is given in Batchelor [p.11] but for simplicity we shall just consider a rectangular element in two dimensions and apply Newton's law of conservation of angular momentum to this element. From figure 1.4, the rate of change of angular

[2] The same idea can be used to define a first rank tensor (or vector) which satisfies $a'_i = t_{i\alpha}a_\alpha$ and can easily be extended to tensors of third and higher rank.

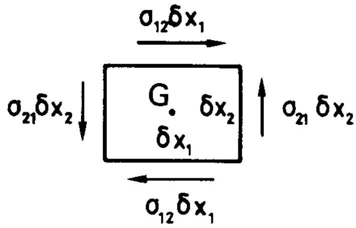

Fig. 1.4. Angular momentum of a fluid element

momentum about the centre of gravity G will be, to lowest order,

$$2(\sigma_{21}\delta x_2)\frac{\delta x_1}{2} - 2(\sigma_{12}\delta x_1)\frac{\delta x_2}{2}$$

where σ_{21} and σ_{12} are evaluated at G. Letting the rectangle shrink to zero and assuming the angular acceleration remains finite implies that

$$\sigma_{21} = \sigma_{12}.$$

This argument can be used in any plane and it follows that

$$\sigma_{ij} = \sigma_{ji}$$

and so the stress tensor is *symmetric*.

It is now convenient to separate σ_{ij} into two components: an *isotropic* part $-p\delta_{ij}$ as would exist in an inviscid fluid and a *deviatoric* part d_{ij} which is due to the viscous forces in the fluid. Thus we write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \tag{1.6}$$

and we are now concerned with modelling the deviatoric stress. From the experimental observations mentioned in §1.1, we assert that d_{ij} varies linearly with the imposed velocity and inversely with the length scale of the apparatus. This leads us to infer that d_{ij} will be a linear function of the velocity gradients $\frac{\partial u_\alpha}{\partial x_\beta}$. This is the assumption that defines a *Newtonian Fluid*. It can be shown that $\left\{\frac{\partial u_\alpha}{\partial x_\beta}\right\}$ satisfies the linear transformation rule (1.5) and it is therefore a tensor (exercise 1). The assumption above implies that there exists a linear relation of the form

$$d_{ij} = A_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}$$

between the deviatoric stress tensor and the tensor $\frac{\partial u_\alpha}{\partial x_\beta}$. Here $A_{ij\alpha\beta}$ is a tensor of rank 4 and, because there is no physically preferred direction in either d_{ij} or $\frac{\partial u_\alpha}{\partial x_\beta}$, it must be an *isotropic* tensor. That is to say it must have the same components in all sets of rotated Cartesian axes so

$$A_{ij\alpha\beta}t_{il}t_{jm}t_{\alpha\gamma}t_{\beta\delta} = A_{lm\gamma\delta} \tag{1.7}$$

for *all* orthogonal transformations t_{il}. The symmetry of σ_{ij} implies that

$$A_{ij\alpha\beta} = A_{ji\alpha\beta} \tag{1.8}$$

in addition, and we now need to determine the most general form of $A_{ij\alpha\beta}$ which will satisfy conditions (1.7) and (1.8). It can be shown by tensor methods that condition (1.7) on $A_{ij\alpha\beta}$ is sufficient to reduce the 81 scalar quantities in $A_{ij\alpha\beta}$ to just 3, and the symmetry condition (1.8) affords a further reduction to two so that

$$A_{ij\alpha\beta} = \lambda\delta_{ij}\delta_{\alpha\beta} + \mu\delta_{i\alpha}\delta_{j\beta} + \mu\delta_{i\beta}\delta_{j\alpha}$$

and, correspondingly

$$d_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.9}$$

where λ, μ are scalar quantities.

We now show how this formula for d_{ij} can be deduced in a more straightforward way *without* resorting to the theory of tensors of rank 4. We first note that a fluid moving as a rigid body will experience no stress. Thus σ_{ij} will be zero when $\mathbf{u}$ is either a function of time alone or equal to $\boldsymbol{\omega} \wedge \mathbf{r}$ where $\boldsymbol{\omega}$ is a vector which varies only with time. This observation shows that σ_{ij} will depend only on the components $\frac{\partial u_1}{\partial x_1}$, $\frac{\partial u_2}{\partial x_2}$, $\frac{\partial u_3}{\partial x_3}$, $\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$, $\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$, and $\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}$. We therefore define the *rate of strain tensor*[3] $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and conclude that there exists a linear relation $d_{ij} = B_{ij\alpha\beta}e_{\alpha\beta}$ between the two *symmetric* tensors $d_{ij}, e_{\alpha\beta}$. This can be written more simply as

$$D = P_1e_{11} + P_2e_{22} + P_3e_{33} + Q_1e_{23} + Q_2e_{31} + Q_3e_{12} \tag{1.10}$$

where D is the matrix $[d_{ij}]$ and P_i, Q_i are symmetric 3×3 matrices. Thus there are now 36 unknown scalar quantities contained in P_i and Q_i. Now, rather than considering a general transformation, we consider a particular transformation T_1 which rotates the axes through $\frac{\pi}{2}$ about the x_1 axis. Then

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = T_1\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} x_1 \\ x_3 \\ -x_2 \end{pmatrix} \tag{1.11}$$

[3] Exercise 1 shows that e_{ij} is a tensor.

and the effect of the rotation on e_{ij} is to make

$$\left.\begin{aligned} e'_{11} = e_{11}, \quad e'_{22} = e_{33}, \quad e'_{33} = e_{22}, \\ \text{and} \qquad e'_{23} = -e_{23}, \quad e'_{31} = -e_{12}, \quad e'_{12} = e_{31}. \end{aligned}\right\} \tag{1.12}$$

The isotropy condition (1.7) means that

$$D' = T_1 D T_1^T, \tag{1.13}$$

so using (1.10) and (1.12) leads to

$$\begin{aligned} T_1P_1T_1^Te_{11} \quad &+T_1P_2T_1^Te_{22} + T_1P_3T_1^Te_{33} \\ &+T_1Q_1T_1^Te_{23} + T_1Q_2T_1^Te_{31} + T_1Q_3T_1^Te_{12} \\ &\equiv P_1e_{11} + P_2e_{33} + P_3e_{22} - Q_1e_{23} - Q_2e_{12} + Q_3e_{31}. \end{aligned} \tag{1.14}$$

Equating the coefficients of e_{11}, gives $P_1 = T_1P_1T_1^T$ and this shows (exercise 2) that $P_1 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}$ for some scalar α, β. By symmetry (or by considering rotations T_2, T_3 about x_2, x_3 axes) we can deduce that $P_2 = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ and $P_3 = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ and this makes the coefficients of e_{22} and e_{33} in (1.14) identical automatically. Similarly the coefficients of e_{23} in (1.14) are identical if $Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & \delta \\ 0 & \delta & -\gamma \end{pmatrix}$ and using symmetry and equating coefficients of e_{31} and e_{12} in (1.14) shows further that $\gamma = 0$. Thus we get

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta \\ 0 & \delta & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & \delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and we have reduced (1.10) to a form which depends on only three scalar quantities.

We have so far considered only rotations through $\frac{\pi}{2}$ to achieve this great simplification but we can reduce the number of unknowns still further by considering a rotation through some other angle. We take $T_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and evaluate d'_{11} by the two expressions in

(1.13). From the left-hand side we get

$$\begin{aligned} d'_{11} &= (T_\theta D T_\theta^T)_{11} = d_{11}\cos^2\theta + 2d_{12}\sin\theta\cos\theta + d_{22}\sin^2\theta \\ &= (\alpha e_{11} + \beta e_{22} + \beta e_{33})\cos^2\theta \\ &\quad +2\delta e_{12}\sin\theta\cos\theta + (\beta e_{11} + \alpha e_{22} + \beta e_{33})\sin^2\theta \end{aligned}$$

and from the right-hand side

$$\begin{aligned} d'_{11} &= \alpha e'_{11} + \beta e'_{22} + \beta e'_{33} \\ &= \alpha(e_{11}\cos^2\theta + 2e_{12}\cos\theta\sin\theta + e_{22}\sin^2\theta) \\ &\quad +\beta(e_{11}\sin^2\theta - 2e_{12}\cos\theta\sin\theta + e_{22}\cos^2\theta) + \beta e_{33}. \end{aligned}$$

Finally, equating these two expressions for arbitrary θ leads to $\delta = \alpha - \beta$.

Thus the expression for d_{ij} can now be written in tensor form as

$$d_{ij} = \beta\delta_{ij}e_{kk} + \delta e_{ij}$$

and it can easily be checked that this is an isotropic tensor which transforms according to the rule (1.5) for *any* transformation t_{ij}. Finally, writing $\delta = 2\mu$ and $\beta = \lambda$ we recover (1.9). Therefore, having assumed a linear dependence of d_{ij} on $\frac{\partial u_\alpha}{\partial x_\beta}$, the most general expression for the stress tensor is

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad (1.15)$$

This is the constitutive equation for a *Newtonian Viscous Fluid* which depends on two scalar parameters μ, the *dynamic shear viscosity* and λ, the *bulk viscosity*. The former measures the response of the fluid to shearing and extension; λ measures the response to changes of volume and is irrelevant for the incompressible flows considered in these notes. These two quantities may depend on the local temperature, density, or pressure of the fluid but, in these notes, we will not allow them to depend on $\mathbf{u}$.

1.3 The Navier-Stokes equations

We are now in a position to apply the above ideas and formulate a definitive model for the motion of a viscous fluid. The only mathematical tool we need is the *Transport Theorem* which states that if $V(t)$ is a region which always contains the same fluid particles then

$$\frac{d}{dt}\left[\int\int\int_{V(t)} F(\mathbf{x},t)dv\right] = \int\int\int_{V(t)}\left(\frac{dF}{dt} + F\nabla.\mathbf{u}\right)dv \qquad (1.16)$$

where the derivatives of F exist and $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}.\nabla$ is the convective derivative. This is a relatively simple example of differentiation under the integral sign (exercise 4).

1.3.1 Conservation of mass

By taking F as the density, ρ, in the above theorem (1.16), the principle of conservation of mass for any volume $V(t)$ leads immediately to the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0, \tag{1.17}$$

on assuming suitable smoothness for $\rho, \mathbf{u}$ (e.g. differentiability in space and time). When ρ is constant, the fluid is said to be incompressible[4] and this equation reduces to

$$\nabla.\mathbf{u} = 0. \tag{1.18}$$

We shall assume that ρ is constant throughout the rest of this book unless specifically stated otherwise.

1.3.2 Conservation of momentum

Applying Newton's law to the fluid in $V(t)$, the rate of change of the momentum, $\int_V \rho \mathbf{u} dV$, equals the force exerted on the fluid in V. In the absence of body forces, like gravity or electromagnetic effects, the only forces acting on V are the viscous forces exerted on the boundary ∂V by the surrounding fluid. Using (1.3), the force in the x_i direction can be written as

$$\int\int_{\partial V} \sigma_{ij} n_j dS = \int\int\int_V \frac{\partial}{\partial x_j}(\sigma_{ij}) dV$$

by Green's theorem. If we now apply the transport theorem to this momentum balance and use equation (1.18) we see that

$$\rho \frac{du_i}{dt} = \frac{\partial}{\partial x_j}(\sigma_{ij}) \tag{1.19}$$

or, substituting for σ_{ij} from (1.15) and using (1.18) again,

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right). \tag{1.20}$$

[4] Note that some scientists, e.g., oceanographers, use $\frac{d\rho}{dt} = 0$ to imply incompressibility, but (1.18) still applies.