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Wavelets and Operators

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"Ce à quoi l'un s'était failli, l'autre est arrivé et ce qui était inconnu à un siècle, le siècle suivant l'a éclairci, et les sciences et les arts ne se jettent pas en moule mais se forment et figurent en les maniant et polissant à plusieurs fois [...] Ce que ma force ne peut découvrir, je ne laisse pas de le sonder et essayer et, en retastant et pétrissant cette nouvelle matière, la remuant et l'eschaufant, j'ouvre à celui qui me suit quelque facilité et la lui rends plus souple et plus maniable. Autant en fera le second au tiers qui est cause que la difficulté ne me doit pas désespérer, ni aussi peu mon impuissance ..."

Montaigne, *Les Essais*, Livre II, Chapitre XII.

"Where someone failed, another has succeeded; what was unknown in one century, the next has discovered; science and the arts do not grind themselves into uniformity, but gain shape and regularity by carving and polishing repeatedly [...] What my own strength has not been able to uncover, I cease not from working at and trying out and, by re-shaping and solidifying this new material, in moulding and heating it, I bequeathe to him who follows some facility and make it the more supple and malleable for him. The second will do the same for the third, which is why difficulty does not make me despair, nor of my own weakness..."

Preface to the English edition

The "Theory of Wavelets" lies on the boundaries between (1) mathematics (2) scientific calculation (3) signal processing and (4) image processing. The aim of the theory is to give a coherent set of concepts, methods and algorithms to deal with the difficulties met in each of these disciplines.

Wavelet analysis has come to have applications in widely differing areas of science, because such analysis gives information (of "time–scale" type) about certain signals, images and operators which is more pertinent than that obtained from standard Fourier analysis (or the "time–frequency" methods that derive from it).

The present book has been written by a mathematician and is intended primarily for mathematicians, without forgetting statisticians and engineers working on signal- and image-processing. Our intention has been to take the greatest care in presenting the various constructions of wavelets, and to describe their utilisation within mathematics. The reader whose interest lies in applications of the "wavelet machine" outside mathematics is invited to turn to the second part of the bibliography, which has been specially prepared for the English edition.

I am very conscious of the honour of being published by the Cambridge University Press and I have admired the quality of David Salinger's translation.

Yves Meyer, Paris, 3rd November 1991.

Introduction

For many years, the sine, cosine and imaginary exponential functions have been the basic functions of analysis. The sequence $(2\pi)^{-1/2}e^{ikx}$, $k = 0, \pm1, \pm2, \ldots$ forms an orthonormal basis of the standard space $L^2[0, 2\pi]$; Fourier series are the linear combinations $\sum a_k e^{ikx}$. Their study has been and remains, an unquenchable source of problems and discoveries in mathematical analysis. The problems arise from the absence of a good dictionary for translating the properties of a function into those of its Fourier coefficients. Here is an example of the kind of difficulty that occurs. J.P. Kahane, Y. Katznelson and K. de Leeuw have shown ([150]) that, to get a continuous function $g(x)$ from an arbitrary square-summable function $f(x)$, it is sufficient to increase—or leave unchanged—the moduli of the Fourier coefficients of $f(x)$ and to adjust their phases judiciously. It is thus impossible to predict the properties (size, regularity) of a function solely from knowledge of the order of magnitude of its Fourier coefficients. Indeed it is still difficult if we know the Fourier coefficients explicitly, and many problems are still open.

At the beginning of the 1980s, many scientists were already using "wavelets" as an alternative to traditional Fourier analysis. This alternative gave grounds for hoping for simpler numerical analysis and more robust synthesis of certain transitory phenomena. The "wavelets" of J.S. Liénard or of X. Rodet ([167], [206]) were used for numerical treatment of acoustic signals (words or music) and those of J. Morlet ([124]) for stocking and interpreting seismic signals gathered in the course of oil prospecting expeditions. Among mathematicians, research was just as active: to mention only the most striking, R. Coifman and G. Weiss ([75]) invented the "atoms" and "molecules" which were to form the basic building blocks of various function spaces, the rules of assembly being clearly defined and easy to use. Certain of these atomic decompositions could, moreover, be obtained by making a discrete version of a well-known identity, due to A. Calderón, in which "wavelets" were implicitly involved. That identity was later rediscovered by Morlet and his collaborators .... Lastly, L. Carleson used functions very similar to "wavelets" in order to construct an unconditional basis of the H^1 space of E.M. Stein and G. Weiss.

These separate investigations had such a "family resemblance" that it seemed necessary to gather them together into a coherent theory, mathematically well-founded and at the same time, universally applicable. The **orthonormal wavelet bases**, whose construction is given in the present volume, are a replacement for the empirical "wavelets" of Liénard, Morlet and Rodet.

The same orthonormal wavelet bases give direct access to the "atomic decompositions" of Coifman and Weiss, which are thus—for the first time—related to constructions of **unconditional bases** of the standard spaces of functions and distributions. The wavelet bases are universally applicable: "everything that comes to hand", whether function or distribution, is the sum of a wavelet series and, contrary to what happens with Fourier series, the coefficients of the wavelet series translate the properties of the function or distribution simply, precisely and faithfully.

So we have a new tool at our command, an instrument that lets us perform, without thinking, the delicate constructions that could not formerly be achieved without recourse to lacunary, or random, Fourier series. The exceptional properties of the sums of these special series become the everyday properties of generic sums of wavelet series.

The algorithms for analysis by, and synthesis of, orthogonal wavelet series will, doubtless, play an important role in many different branches of science and technology. Mathematicians, physicists, and engineers who want to know everything about wavelets now have the present volume (Chapters 1–6) of this work at their disposal.

Volumes 2 and 3 are addressed more specifically to an audience of mathematicians. They deal with the operators associated with wavelets. G. Weiss has shown that the study of the operators acting on a space of functions or distributions can become very simple when the elements of the space admit "atomic decompositions". He writes "many problems in analysis have natural formulations as questions of continuity of linear operators defined on spaces of functions or distributions. Such questions can often be answered by rather straightforward techniques if they can first be reduced to the study of the operator on an appropriate class of simple functions which, in some convenient sense, generate the entire space." When these "simple elements" were the functions e^{ikx} of the trigonometric system, the bounded operators T on L^2, which were diagonal with respect to the trigonometric system, did not have any other interesting property (with the exception of translation-invariance, which follows immediately from the definition). It was then necessary to impose quite precise conditions on the eigenvalues of T in order to extend such an operator to other function spaces: the first results in this direction were obtained by J. Marcinkiewicz.

However, the bounded operators which can be diagonalized exactly or approximately, with respect to the wavelet basis, form an algebra $\mathbf{A}$ of bounded operators on L^2 and the well-known **Calderón-Zygmund real-variable methods** enable the operators of $\mathbf{A}$ to be extended to other spaces of functions or distributions. The algebra $\mathbf{A}$, which extends the pseudo-differential operators in a natural sense, is strictly contained in the set $\mathbf{C}$ of operators whose study has been recommended by Calderón. Work on these operators should enable us to solve several outstanding classical problems in complex analysis and partial differential equations.

Here is a slightly more precise description of the set $\mathbf{C}$, the delicate construction of which we have called "Calderón's programme". After having invented, together with A. Zygmund, what was to become the classical pseudo-differential calculus, Calderón intended to extend the field of application systematically, by weakening, as far as possible, the regularity hypotheses necessary for the algorithms to work.

The fundamental—and unexpected—discovery made by Calderón was the existence of a limit to the search for minimal hypotheses of regularity. There is a "natural boundary" which cannot be transgressed, and the extension of operators to this boundary is precisely the analytic extension of holomorphic functions on certain Banach spaces, as we shall show in Chapter 13.

Chapters 7–9 are devoted to the study and then the construction of the set $\mathbf{C}$ of operators of Calderón's programme. We call them the **Calderón-Zygmund operators**, although they are very different from the "historical operators" studied by Calderón and Zygmund in the 1950s and 60s.

Just like these "historical operators", those we consider can be defined by singular integrals, in a new sense which we clarify in Chapter 7. To go beyond the context of convolution operators, it becomes indispensable to have a criterion for L^2-continuity, without which the theory collapses like a house of cards. One such criterion is the well-known theorem $T(1)$ due to David and Journé, which we shall prove in Chapter 8. Theorem $T(1)$ replaces the Fourier transform, whose use remains restricted to convolution operators.

Unfortunately, theorem $T(1)$, although giving a necessary and sufficient condition, is not directly applicable to the most interesting operators of the set **C** of Calderón's programme. We do not know why that is. The operators in question have, however, a very special non-linear structure, which, when correctly exploited, allows us to pass from the "local" results given by David and Journé's theorem to the "global" theorems necessary for the functioning of Calderón's programme.

In Volume 3 of this book (Chapters 12–16) and also in Chapter 9 of Volume 2, we have given the most beautiful of the applications of Calderón's programme. First comes the celebrated pseudo-differential calculus, initiated by Calderón, which, at present, has interesting and important applications to non-linear partial differential equations.

Then we pass to complex analysis and the Hardy spaces associated with Lipschitz domains of the complex plane. The object of Chapter 12 is the study of the Cauchy operator on rectifiable curves. We then examine the problem, posed by T. Kato, of determining the domain of the square roots of second order pseudo-differential operators, in the accretive case.

After that, we give an account of the results of B. Dahlberg, D. Jerison, C. Kenig and G. Verchota relative to the Dirichlet and Neumann problems in Lipschitz open sets.

The book ends with a brief presentation of J.M. Bony's paradifferential operators, which serve to analyse the regularity of non-linear partial differential operators.

Wavelets reappear, in a surprising way, as the eigenfunctions of certain paradifferential operators. Correctly handled, they remain present in the study of Hardy spaces and the Cauchy operator on a Lipschitz curve: the operator is "almost diagonal" with respect to a wavelet basis specially designed for complex analysis on that curve (Chapter 11). The construction of wavelet bases is thus sufficiently supple to be adaptable to differing geometric situations: we also obtain "non-orthogonal wavelet bases". At present, there is no universal basis which can simultaneously be used in the analysis of all the operators of Calderón's class **C**.

J.O. Strömberg was the first to construct an orthonormal basis of $L^2(\mathbb{R})$, of the form $2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, where, for each $m \in \mathbb{N}$, the function $\psi(x)$ was of class C^m and decreased exponentially at infinity.

Subsequent work on orthogonal wavelets has not followed the path discovered by Strömberg. That work is, essentially, due to I. Daubechies, P.G. Lemarié, S. Mallat, and the author. It is given here, with care and with complete proofs.

As far as operators are concerned, the well-known results of Calderón, Zygmund, and Cotlar will be described in Chapter 7, in the new context of the set **C**.

The other names the reader of this work will often encounter are J.M. Bony, G. David, P. Jones, J.L. Journé, C. Kenig, T. Murai, and S. Semmes.

The division into three volumes will allow this work to be read in several ways. As we have already suggested, the reader may wish to go no further than the present volume, which surveys our present knowledge about wavelets. But Volume 2 (Calderón-Zygmund operators) may also be read directly, assuming only the results quoted in the introductions of the first six chapters. Finally, the reader can go straightaway to Chapters 12, 13, 14, 15 or 16 of Volume 3, because each of them forms a coherent account of a subject of independent interest (complex analysis, holomorphic functionals on Banach spaces, Kato theory, elliptic partial differential equations in Lipschitz domains, and, lastly, non-linear partial differential equations). The thread linking these different themes is, quite clearly, the use of wavelets in Calderón's programme in operator theory.

This book has been written at a level appropriate for first-year postgraduates, and we have tested it in France, and in the USA, on various audiences of mathematicians and engineers. To read this book, it is therefore not necessary to have studied the remarkable treatises by E. Stein and G. Weiss ([221)], E. Stein ([217]), or Garcia Cuerva and Rubio de Francia ([115]), not to mention the fundamental text and reference by Zygmund ([239]).

R. Coifman helped me to recognize the importance of Calderón's programme. Since the summer of 1974, our scientific collaboration has been devoted to its realization, and this book has been one of our projects. If our own work no longer appears in its original form in this book, it is because our zeal and enthusiasm have communicated themselves to younger research workers, who have found more elegant solutions to the problems we had been determined to resolve.