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Fourier series and integrals, filtering and sampling

1 Introduction

Wavelet series provide a simpler and more efficient way to analyse those functions and distributions that have hitherto been studied by means of Fourier series and integrals. But wavelet analysis cannot entirely replace Fourier analysis, indeed, the latter is used in constructing the orthonormal bases of wavelets needed for analysis with wavelet series. To construct the basic wavelets we use what "works well" in Fourier analysis—its algebraic formalism. Once the wavelets have been constructed, they perform incredibly well in situations where Fourier series and integrals involve subtle mathematics or heavy numerical calculations.

The two kinds of analysis are thus complementary rather than competing. What is more, the reader will need to know the rudiments of Fourier analysis in order to proceed!

In the first chapter we restate the classical formulas and theorems of Fourier analysis, but the chapter also serves as an "overture" in which the "themes" of this book—wavelets and operators—make a preliminary appearance.

2 Fourier series

Fourier series are used to analyse periodic functions or distributions. To begin with, we shall consider the case of one real variable and suppose the period is 2π.

We start with what works best. Let H be the Hilbert space $L^2[0,2\pi]$ with the inner product $(f,g) = \int_0^{2\pi} f(x)\bar{g}(x)\,dx$. Then the functions $(2\pi)^{-1/2}e^{ikx}$, $k \in \mathbb{Z}$, constitute a Hilbert basis of H. With a slight change of normalization, we define the Fourier coefficients of $f \in H$ by

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx, \tag{2.1}$$

and we have

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}, \tag{2.2}$$

the series converging in H.

The identity (2.2) automatically defines an extension of the function $f \in L^2[0,2\pi]$ to the whole real line. This extension is just the 2π-periodic function whose restriction to $[0,2\pi]$ is exactly $f(x)$. The interval $[0,2\pi)$ acts as a fundamental domain for the discrete subgroup $2\pi\mathbb{Z} \subset \mathbb{R}$ and the 2π-periodic functions which are locally square-summable are canonically identified with functions of $L^2[0,2\pi]$. The same remark applies to the spaces $L^p[0,2\pi]$ which we shall meet later. But it cannot apply to the space $F \subset \mathcal{D}'(\mathbb{R})$ of 2π-periodic distributions. Recall that $\mathcal{D}(\mathbb{R})$ is the space of infinitely differentiable functions of compact support and that $\mathcal{D}'(\mathbb{R})$ is the space of continuous linear functionals on $\mathcal{D}(\mathbb{R})$. A distribution $S \in \mathcal{D}'(\mathbb{R})$ is 2π-periodic if $\langle S,u\rangle = \langle S,v\rangle$ whenever u and v belong to $\mathcal{D}(\mathbb{R})$ and satisfy $v(x) = u(x-2\pi)$. Here we have used $\langle\cdot,\cdot\rangle$ to denote the bilinear form which implements the duality between distributions and test functions. 2π-periodic distributions are not characterized by their restrictions to the open interval $(0,2\pi)$ because this process can lose information. One example of such loss involves the "Dirac comb" defined by

$$S = \sum_{-\infty}^{\infty} \delta_{2k\pi} \tag{2.3}$$

where δ_a denotes the Dirac measure at a. The restriction of S to the open interval $(0,2\pi)$ is zero. Similarly, S cannot sensibly be restricted to the closed interval $[0,2\pi]$. It appears, then, that we cannot use the usual formulas to define the Fourier coefficients of a 2π-periodic distribution. We get round this difficulty in the following way. Let E be the vector space of infinitely differentiable 2π-periodic functions. Put E and F in duality by setting

$$\langle S,f\rangle = \langle S,\phi f\rangle \tag{2.4}$$

where $S \in F$, $f \in E$ and where $\phi \in \mathcal{D}(\mathbb{R})$ is such that

$$\sum_{-\infty}^{\infty} \phi(x + 2k\pi) = 1. \tag{2.5}$$

In some sense, such a function ϕ imitates the characteristic function of $[0, 2\pi)$. The left-hand side of (2.4) is defined by the right-hand side, which makes sense because the product ϕf is in $\mathcal{D}(\mathbb{R})$.

It is an exercise to verify the invariance of (2.4) given different choices of ϕ: subtraction of two such choices leads to using the equivalence, for $g \in \mathcal{D}(\mathbb{R})$, of

$$\sum_{-\infty}^{\infty} g(x + 2k\pi) = 0 \tag{2.6}$$

and

$$\exists h \in \mathcal{D}(\mathbb{R}) \qquad \text{such that} \qquad g(x) = h(x + 2\pi) - h(x). \tag{2.7}$$

As a particular case of (2.4) we shall define the Fourier coefficients c_k, $k \in \mathbb{Z}$, of an arbitrary 2π-periodic distribution S. We put $e_k(x) = e^{ikx}$, $\bar{e}_k(x) = e^{-ikx}$ and

$$c_k = \frac{1}{2\pi} \langle S, \bar{e}_k \rangle. \tag{2.8}$$

Then, for some integer $m \in \mathbb{N}$ and constant C, $|c_k| \leq C(1 + |k|)^m$ and this property characterizes the Fourier coefficients of a 2π-periodic distribution. Finally

$$S = \sum_{-\infty}^{\infty} c_k e_k \tag{2.9}$$

and the series on the right-hand side converges in the sense of distributions. This means that, if $u \in \mathcal{D}(\mathbb{R})$ is a test function, then

$$\langle S, \bar{u} \rangle = \sum_{-\infty}^{\infty} c_k \bar{d}_k \qquad \text{where} \qquad d_k = \int_{-\infty}^{\infty} u(x) e^{-ikx}\, dx \tag{2.10}$$

and the series in (2.10) is absolutely convergent, because the d_k decrease rapidly at infinity.

One can similarly use the duality between E and F, instead of that between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$, to interpret (2.9).

If $f \in E$, that is, if f is infinitely differentiable and 2π-periodic, we denote its Fourier coefficients by d_k, $k \in \mathbb{Z}$ (defined as in (2.1)) and we get

$$\langle S, \bar{f} \rangle = 2\pi \sum_{-\infty}^{\infty} c_k \bar{d}_k\,, \tag{2.11}$$

which is Plancherel's identity.

We can differentiate equation (2.9), term by term, arbitrarily often, to give the Fourier series of the derivatives of S (in the sense of distributions).

Let us give two classical applications of this rule.

We start with the "saw-tooth" function $s(x)$ which is odd, 2π-periodic and equal to $(\pi-x)/2$ on $[0,2\pi)$. This function is in $L^2[0,2\pi]$ and satisfies

$$s(x) = \sum_1^\infty \frac{1}{n}\sin nx. \tag{2.12}$$

Before differentiating (2.12) term by term, it is worthwhile drawing the graph of the 2π-periodic function $s(x)$ which highlights the discontinuities of the first kind at each point $2k\pi$, $k \in \mathbb{Z}$.

Now let us differentiate term by term, in the sense of distributions in $\mathcal{D}'(\mathbb{R})$. The derivative of $s(x)$ is the sum of the usual derivative (the constant function $-1/2$) and of πS, where S is the Dirac comb. The series $\sum_1^\infty \cos nx$ appears on the right-hand side and we finally arrive at the result

$$\sum_{-\infty}^{\infty} \delta_{2k\pi} = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{ikx}, \tag{2.13}$$

which is the Poisson summation formula.

Here is another amusing application of the same ideas. We start with the even 2π-periodic function $c(x) = \log(|\sin(x/2)|^{-1})$, where log denotes the natural logarithm. Once again, $c(x)$ belongs to $L^2[0,2\pi]$ and its Fourier series is

$$\log\frac{1}{|\sin(x/2)|} = \log 2 + \sum_1^\infty \frac{1}{n}\cos nx. \tag{2.14.}$$

This calls for some preliminary remarks which will be developed in Chapter 6. Let $H : L^2[0,2\pi] \longrightarrow L^2[0,2\pi]$ denote the operator defined by $H(e^{ikx}) = -i\,\mathrm{sgn}(k)e^{ikx}$ with $H(1) = 0$. In other words, $H(\cos kx) = \sin kx$, for $k \in \mathbb{N}$, and $H(\sin kx) = -\cos kx$, for $k \geq 1$.

The operator H is called the Hilbert transform. It is clearly continuous on $L^2[0,2\pi]$, but is certainly not continuous on $L^\infty[0,2\pi]$. Indeed, the function $s(x)$ belongs to $L^\infty[0,2\pi]$, but the function $c(x)$ is not in L^∞, yet these two functions are related by the identity $(H(s))(x) = -c(x) + \log 2$. We shall discover the explanation of this phenomenon in Chapter 7, when we study the (L^∞, BMO)-continuity of the Calderon-Zygmund operators, of which H is the prototype.

We can make a further remark about (2.14). Just as the derivative of $\log|x|$ is the distribution $\mathrm{PV}\,x^{-1}$, the derivative—in the sense of distributions—of $\log(|\sin(x/2)|^{-1})$ is $\frac{1}{2}\,\mathrm{PV}\cot(x/2)$, a 2π-periodic distribution whose singularities are at $2k\pi$, $k \in \mathbb{Z}$, and are of the same type as those of $\mathrm{PV}\big(1/(x-2k\pi)\big)$. In fact, in a neighbourhood of $2k\pi$, the difference between the two distributions is an infinitely differentiable function.

On differentiating (2.14) in the sense of distributions we get

$$\text{(2.15)} \qquad \mathrm{PV}\cot\frac{x}{2} = \sum_{-\infty}^{\infty} -i\,\mathrm{sgn}(k)e^{ikx} = \sum_{-\infty}^{\infty} H(e^{ikx}).$$

This new identity has a remarkable interpretation: the operator H can be defined as convolution with the distribution $S = (1/2\pi)\,\mathrm{PV}\cot(x/2)$, where the convolution product $S * T$ of two 2π-periodic distributions $S = \sum_{-\infty}^{\infty} c_k e^{ikx}$ and $T = \sum_{-\infty}^{\infty} d_k e^{ikx}$ is defined by

$$S * T = 2\pi \sum_{-\infty}^{\infty} c_k d_k e^{ikx}.$$

This definition extends the usual convolution $f * g$ of two functions in $L^1[0,2\pi]$. Recall that the convolution product is defined by

$$\text{(2.16)} \qquad (f * g)(x) = \int_0^{2\pi} f(x-y)g(y)\,dy$$

where, on the right-hand side, f is extended to the whole of $\mathbb{R}$ by periodicity. The vector space F of 2π-periodic distributions is a topological algebra under the convolution product and the identity of this algebra is the Dirac comb, $\sum_{-\infty}^{\infty} \delta_{2k\pi} = (1/2\pi)\sum_{-\infty}^{\infty} e^{ikx}$.

Finally, for $f \in E$, we write $f(x) = \sum_{-\infty}^{\infty} \alpha_k e^{ikx}$ and get

$$H(f) = \sum_{-\infty}^{\infty} -i\,\mathrm{sgn}\,k\,\alpha_k e^{ikx} = \frac{1}{2\pi}\,\mathrm{PV}\cot\left(\frac{x}{2}\right) * f$$
$$= \frac{1}{2\pi}\lim_{\varepsilon\downarrow 0}\int_{\varepsilon\le|y|\le\pi} f(x-y)\cot\frac{y}{2}\,dy.$$

What we have here is the realization of the operator H as a singular integral: this point of view will be developed systematically in the course of Chapter 7.

Fourier analysis in the $L^p[0,2\pi]$ spaces is subtler than that in E, F and $L^2[0,2\pi]$.

When $1 < p < \infty$, the partial sums $\sum_{-N}^{N} c_k e^{ikx}$ of the Fourier series of a function $f \in L^p[0,2\pi]$ converge to f in L^p norm. This result follows from the continuity of the Hilbert transform H on $L^p[0,2\pi]$ when $1 < p < \infty$. The Hilbert transform is not continuous on L^1 or on the space of continuous 2π-periodic functions on $\mathbb{R}$. As a result, the theorem about norm-convergence of partial sums is no longer true when $p = 1$ or $p = \infty$.

The L^p norm of f cannot be evaluated just from the amplitudes of the

Fourier coefficients of f: the phases of the coefficients play an essential part. Here are two examples.

If $\sum_{-\infty}^{\infty} |c_k|^2 < \infty$, then, for almost all choices of signs $\pm$ (in the sense of independent, centred, Bernoulli random variables), the random Fourier series $\sum_{-\infty}^{\infty} \pm c_k e^{ikx}$ converges to a function that, for $2 \le p < \infty$, belongs to each L^p space. But the condition $\sum_{-\infty}^{\infty} |c_k|^2 < \infty$ is sufficient for $\sum_{-\infty}^{\infty} c_k e^{ikx}$ to belong to L^p only when $p = 2$. The random choice of signs $\pm$ has let us scale the ramparts of the L^p spaces.

Consider the particular series $\sum_1^{\infty} k^{-\alpha} e^{ikx}$ for $1/2 < \alpha < 1$. Its sum is a 2π-periodic function $f_\alpha(x)$ whose restriction to $[-\pi, \pi]$ is continuous except at the origin. If x tends to 0 from above, $f(x)$ is equivalent to $c(\alpha)x^{-1+\alpha}$, where $c(\alpha)$ is a non-zero complex constant. A necessary and sufficient condition for $f_\alpha(x)$ to belong to $L^p[0, 2\pi]$ is that $p(1-\alpha) < 1$. But the corresponding random series $\sum_1^{\infty} \pm k^{-\alpha} e^{ikx}$ define (almost everywhere) functions which are continuous (and which belong to the Hölder class of exponent β for $\beta < \alpha - 1/2$). The well-known Brownian motion can be described by such random Fourier series ([149]).

When we want to calculate the L^p norm of a function, knowing each Fourier coefficient gives only the illusion of precision. In calculating every Fourier coefficient, we have taken the analysis too far. We must retrace our steps a little and group the coefficients into so-called dyadic blocks, which we do not break down any further. The dyadic blocks of the Fourier series of f are

$$\Delta_j f(x) = \sum_{2^j \le |k| < 2^{j+1}} c_k e^{ikx}, \qquad j \in \mathbb{N}, \tag{2.17}$$

and the fundamental result of Littlewood and Paley ([171]) is that, for $1 < p < \infty$, the two norms $||f||_p$ and $|c_0| + ||(\sum_0^{\infty} |\Delta_j f(x)|^2)^{1/2}||_p$ are equivalent. This means that changes of sign, which could not previously be made without changing the L^p status of the functions, become quite harmless when applied to the dyadic blocks $\Delta_j f$. If $\sum_0^{\infty} \Delta_j f$ belongs to L^p then so does $\sum_0^{\infty} \varepsilon_j \Delta_j f$, for every choice of $\varepsilon_j = \pm 1$.

Decomposing a Fourier series into dyadic blocks plays just as essential a role in the analysis of the Hölder spaces C^α. But we then need to define the operators Δ_j more carefully. To do this, we take a function, $\psi(x) \in \mathcal{D}(\mathbb{R})$, which is even, supported by the set $1/2 \le |x| \le 3/2$, and such that $1 = \psi(x) + \psi(x/2) + \psi(x/4) + \cdots$, when $|x| \ge 1$. We then change the definition of the Δ_j to

$$\Delta_j f(x) = \sum \psi(k2^{-j}) c_k e^{ikx}. \tag{2.18}$$

Then, for every $\alpha > 0$, f is in C^α if and only if $||\Delta_j f||_\infty = O(2^{-j\alpha})$ as $j \to \infty$. Recall that, for $0 < \alpha < 1$, the Hölder space C^α is composed of those continuous 2π-periodic functions whose moduli of continuity $\omega(h)$ are $O(h^\alpha)$. When $\alpha = 1$, $||\Delta_j(f)||_\infty = O(2^{-j})$ means that $f \in \Lambda_\star$ and not that $f \in C^1$ (the Zygmund class, $\Lambda_\star$, is the Banach space of continuous 2π-periodic functions $f(x)$ satisfying $|f(x+y)+f(x-y)-2f(x)| \le Cy$, for each real x and $0 \le y \le 1$). For $1 < \alpha \le 2$, the C^α spaces, are subspaces of (the usual) C^1 and are defined by the condition $f' \in C^{\alpha-1}$ (where f' is the derivative of f). The definition extends similarly to all $\alpha > 0$.

The partial sums of the Fourier series of a C^α function do not, in general, tend to that function in C^α and are not even uniformly bounded in C^α norm. The same phenomenon occurs for the C^0 norm (the uniform norm) and the L^1 norm.

The remedy lies in the introduction of summability methods. We give Borel's method as an example. Instead of approximating the series $\sum_{-\infty}^{\infty} c_k e^{ikx}$ by its partial sums $\sum_{-N}^{N} c_k e^{ikx}$, we approximate by $\sum_{-\infty}^{\infty} c_k e^{-|k|\varepsilon} e^{ikx}$ and then let ε tend to zero. This amounts to taking the convolution of the function f, whose Fourier coefficients are the c_k, with an approximate identity P_ε. More precisely, we start by extending $f(x)$ to a 2π-periodic function or distribution and put

$$P_\varepsilon(x) = \frac{1}{\varepsilon} P\left(\frac{x}{\varepsilon}\right) \qquad \text{where} \qquad P(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

Then, if $f(x)$ is continuous (and 2π-periodic), $f * P_\varepsilon$ converges to $f(x)$ uniformly; the same holds if $f(x)$ is locally integrable.

Many other approximate identities are frequently used and any reader who wishes to study them more closely may refer to [239].

3 Fourier integrals

We start by defining the "naive" Fourier integrals: those of functions f in $L^1(\mathbb{R})$. Here we simply put

$$(3.1) \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx \qquad \text{for} \qquad -\infty < \xi < \infty .$$

There is no difficulty in showing that $\hat{f}(\xi)$ is a continuous function of ξ and that this function vanishes at infinity.

The convolution product of two functions f and g in $L^1(\mathbb{R})$, gives a third function h in $L^1(\mathbb{R})$ and we get $\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. The Fourier transforms of $L^1(\mathbb{R})$-functions thus form a subalgebra of $C_0(\mathbb{R})$. This subalgebra is called the Wiener algebra, is denoted by $A(\mathbb{R})$ and becomes a Banach algebra if we put $\|\hat{f}\|_A = \|f\|_1$.

Despite its apparent simplicity, the Wiener algebra has not yet given up all its secrets: the interested reader may refer to [148] or to [209].

A property that follows easily from (3.1) is

$$\int \hat{f}(\xi)g(\xi)\,dx = \int f(x)\hat{g}(x)\,dx. \tag{3.2}$$

Equation (3.2) is a simple consequence of Fubini's theorem applied to $\iint f(x)g(\xi)e^{-ix\xi}\,dx\,d\xi$.

In this book, we shall make systematic use of the following remark:

Lemma 1. *Let $f(x)$ be a function in $L^1(\mathbb{R})$. For $T > 0$, let $g(x) = \sum_{-\infty}^{\infty} f(x+kT)$. Then $g(x) \in L^1[0,T]$ and the Fourier coefficients*

$$c_k = \frac{1}{T}\int_0^T g(x)\exp\left(\frac{-2\pi ikx}{T}\right)\,dx$$

of g are given by

$$c_k = \frac{1}{T}\hat{f}\left(\frac{2k\pi}{T}\right).$$

The lemma follows immediately from Fubini's theorem and from the fact that $[0,T)$ is a fundamental domain for the subgroup $T\mathbb{Z}$ of $\mathbb{R}$.

As an application of Lemma 1, we give a proof—found in a physics book—of the Fourier inversion formula. Suppose that $f(x)$ and $\hat{f}(\xi)$ are continuous, $O(x^{-2})$ and $O(\xi^{-2})$, respectively, as x and ξ tend to $\pm\infty$. Then we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\hat{f}(\xi)\,d\xi. \tag{3.3}$$

To see this, we apply Lemma 1. The function $g(x)$ is continuous, because the series defining it is uniformly convergent. The Fourier coefficients of $g(x)$ are $O(k^{-2})$, so $g(x)$ is the sum of its Fourier series. We thus get

$$\sum_{-\infty}^{\infty} f(x+kT) = \frac{1}{T}\sum_{-\infty}^{\infty}\hat{f}\left(\frac{2k\pi}{T}\right)\exp\left(i\frac{2k\pi}{T}\right). \tag{3.4}$$

Having got this far, we fix x and let T tend to infinity, interpreting the right-hand side of (3.4) as a Riemann sum. This gives (3.3).

The physicists' point of view is to consider a function which is $O(x^{-2})$ at infinity as a periodic function of infinite period!

The assumptions of this argument hold in the special case of f belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$. This means that, for each integer $m \geq 1$ and $n \geq 0$, the n^{th} derivative, $f^{(n)}(x)$, of $f(x)$ is $O(x^{-m})$ as $x \to \pm\infty$. Then simple calculations (term by term differentiation and integration by parts) are enough to show that $\hat{f} \in \mathcal{S}(\mathbb{R})$, whenever $f \in \mathcal{S}(\mathbb{R})$.

So (3.3) holds when $f \in \mathcal{S}(\mathbb{R})$. It follows that $f \in \mathcal{S}(\mathbb{R})$ if and only if $\hat{f} \in \mathcal{S}(\mathbb{R})$ and that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, where $\mathcal{F}$ denotes the Fourier transform, is a topological isomorphism.

We use (3.3) to define the Fourier transform $\hat{f}$ of a tempered distribution f. We decree that, for every function $g \in \mathcal{S}(\mathbb{R})$, the meaning of $\langle \hat{f}, g\rangle$ should be $\langle f, \hat{g}\rangle$. In other words, $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is the transpose of the operator $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

An important special case is that of distributions S of compact support. In this case we can define the function $\hat{S}(\xi)$ directly by $\hat{S}(\xi) = \langle S, \bar{e}_\xi\rangle$, where $e_\xi = e^{ix\xi}$.

The Paley-Wiener theorem gives a characterization of the functions $\hat{S}(\xi)$ which correspond to distributions S whose supports lie in a fixed interval $[-l, l]$, namely, that $\hat{S}(\xi)$ can be extended to the complex plane as an entire function, $F(z)$ satisfying, for each $\varepsilon > 0$, $|F(z)| \le C(\varepsilon)\exp((l+\varepsilon)|z|)$. Further, $F(\xi) = \hat{S}(\xi)$ grows slowly on the real axis: there is an integer m such that $|F(\xi)| \le C(1+|\xi|)^m$.

One example of a tempered distribution is given by a 2π-periodic function, f, whose restriction to $[0, 2\pi]$ is in $L^1[0, 2\pi]$. Here the Fourier transform in the sense of distributions, f, is given by

$$(3.5) \qquad \hat{f} = 2\pi \sum_{-\infty}^{\infty} c_k \delta_k \quad \text{where} \quad c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\, dx$$

and where δ_k is Dirac measure at k.

Let us now give the definition of the Fourier transform, $\mathcal{F}$, on $L^2(\mathbb{R})$. Equations (3.2) and (3.3) imply that $(\hat{f}, \hat{g}) = 2\pi(f, g)$ when f and g belong to $\mathcal{S}(\mathbb{R})$ —we have put $(f, g) = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\, dx$. As a result, $(2\pi)^{-1/2}\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ becomes an isometry when $\mathcal{S}(\mathbb{R})$ is regarded as a subspace of $L^2(\mathbb{R})$. Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the isometry extends to the whole of $L^2(\mathbb{R})$.

For f belonging to $L^2(\mathbb{R})$, the integral $\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx$ will be defined as the limit, in $L^2(\mathbb{R})$ norm, of the truncated integrals $g_T(\xi) = \int_{-T}^{T} e^{-ix\xi} f(x)\, dx$.

Using Carleson's theorem, C. Kenig has shown that the truncated integrals $g_T(\xi)$ converge almost everywhere to $\hat{f}(\xi)$ as $T \to \infty$. We refer the reader to [161].

4. Filtering and sampling

For technological reasons, signals that one wants to analyse have a frequency spectrum limited to a "band" $[-T, T]$. (Recall that mathematical frequencies have a sign which arises from the use of the formulas $\cos \omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$ and $\sin \omega t = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})$, $\omega > 0$.) So it is essential to make full use of the fact that the frequencies contained in a signal do not exceed T. That is the purpose of this section. To begin with, we study the sampling of such signals. That is, we give a rule which lets us decide whether the measurements $f(k\delta)$, $\delta > 0$, $k \in \mathbb{Z}$, of a signal, are sufficient to recover the signal from those measurements when "Shannon's condition" is satisfied.

After this preamble, we come to the precise mathematical statement. For $T > 0$, let $\mathcal{E}_T$ denote the vector space of distributions $f \in \mathcal{S}'(\mathbb{R})$ whose Fourier transforms S have supports in $[-T, T]$.

The Fourier inversion formula gives $\hat{S}(\xi) = 2\pi f(-\xi)$. Now, $\hat{S}(\xi)$ is the restriction to the real axis of an entire function of exponential type, and so the same holds for $f(\xi)$. In particular, f is infinitely differentiable and thus continuous on the real axis, so the sampling $f(k\delta)$, $k \in \mathbb{Z}$, makes sense.

The answer to the basic problem is given by the following result.

Theorem 1. *If $\delta > \pi T^{-1}$, the sampling $f(k\delta)$, $k \in \mathbb{Z}$, does not determine $f \in \mathcal{E}_T$ uniquely, even under the additional assumption that the sequence $f(k\delta)$ decreases rapidly and that the function f belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$.*

If $\delta = \pi T^{-1}$, the sampling $f(k\delta)$, $k \in \mathbb{Z}$, is sufficient to determine $f \in \mathcal{E}_T$ as long as, in addition, $f(k\delta) \in l^p(\mathbb{Z})$, for $1 < p < \infty$, and $f \in L^p(\mathbb{R})$. If $p = \infty$ and if c_k, $k \in \mathbb{Z}$, is a sequence belonging to $l^\infty(\mathbb{Z})$, then a function $f \in \mathcal{E}_T$ belonging to $L^\infty(\mathbb{R})$, such that $f(k\delta) = c_k$, $k \in \mathbb{Z}$, neither necessarily exists nor is necessarily unique. If c_k, $k \in \mathbb{Z}$, belongs to $l^1(\mathbb{Z})$, a function $f \in L^1(\mathbb{R}) \cap \mathcal{E}_T$, with $f(k\delta) = c_k$, $k \in \mathbb{Z}$, may not exist, but is unique when it does.

Lastly, if $0 < \delta < \pi T^{-1}$, let $\phi \in \mathcal{S}(\mathbb{R})$ be a function whose Fourier transform $\hat{\phi}$ vanishes outside $[-\pi\delta^{-1}, \pi\delta^{-1}]$ and equals 1 on the interval $[-T, T]$. Then, for $f \in \mathcal{E}_T$,

$$f(k\delta) = \int_{-\infty}^{\infty} f(x)\bar{\phi}(x - k\delta)\,dx \tag{4.1}$$

and

$$f(x) = \delta \sum_{-\infty}^{\infty} f(k\delta)\phi(x - k\delta). \tag{4.2}$$

The significance of equations (4.1) and (4.2) is that they look as if the functions $\delta^{1/2}\phi(x - k\delta)$, $k \in \mathbb{Z}$, formed an orthogonal basis of the space $V_T = L^2 \cap \mathcal{E}_T$.

But they do nothing of the kind, for three reasons: the functions in question don't lie in V_T but in the slightly larger space $V_{\pi\delta^{-1}}$, they are linearly dependent and they are not mutually orthogonal. In Chapter 2 we shall see how to define an "intermediate space" between V_T and