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# HYDRODYNAMICS

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# HYDRODYNAMICS

BY

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## PREFACE

**T**HIS may be regarded as the sixth edition of a *Treatise on the Mathematical Theory of the Motion of Fluids*, published in 1879. Subsequent editions, largely remodelled and extended, have appeared under the present title.

In this issue no change has been made in the general plan and arrangement, but the work has again been revised throughout, some important omissions have been made good, and much new matter has been introduced.

The subject has in recent years received considerable developments, in the theory of the tides for instance, and in various directions bearing on the problems of aeronautics, and it is interesting to note that the “classical” Hydrodynamics, often referred to with a shade of depreciation, is here found to have a widening field of practical applications. Owing to the elaborate nature of some of these researches it has not always been possible to fit an adequate account of them into the frame of this book, but attempts have occasionally been made to give some indication of the more important results, and of the methods employed.

As in previous editions, pains have been taken to make due acknowledgment of authorities in the footnotes, but it appears necessary to add that the original proofs have often been considerably modified in the text.

I have again to thank the staff of the University Press for much valued assistance during the printing.

HORACE LAMB

*April 1932*

## CONTENTS

*Foreword, by R.A. Caflisch*

xvii

### CHAPTER I

#### THE EQUATIONS OF MOTION

ART.		PAGE
1, 2.	Fundamental property of a fluid . . . . .	1
3.	The two plans of investigation . . . . .	1
4-9.	'Eulerian' form of the equations of motion. Dynamical equations. Equation of continuity. Physical equations. Surface conditions .	2
10.	Equation of energy . . . . .	8
10 a.	Transfer of momentum . . . . .	10
11.	Impulsive generation of motion . . . . .	10
12.	Equations referred to moving axes . . . . .	12
13, 14.	'Lagrangian' form of the equations of motion and of the equation of continuity . . . . .	12
15, 16.	Weber's transformation . . . . .	14
16 a.	Equations in polar co-ordinates . . . . .	15

### CHAPTER II

#### INTEGRATION OF THE EQUATIONS IN SPECIAL CASES

17.	Velocity-potential. Lagrange's theorem . . . . .	17
18, 19.	Physical and kinematical relations of $\phi$ . . . . .	18
20.	Integration of the equations when a velocity-potential exists. Pressure- equation . . . . .	19
21-23.	Steady motion. Deduction of the pressure-equation from the principle of energy. Limiting velocity . . . . .	20
24.	Efflux of liquids; vena contracta . . . . .	23
24 a, 25.	Efflux of gases . . . . .	25
26-29.	Examples of rotating fluid; uniform rotation; Rankine's 'combined vortex'; electromagnetic rotation . . . . .	28

### CHAPTER III

#### IRROTATIONAL MOTION

30.	Analysis of the differential motion of a fluid element into strain and rotation . . . . .	31
31, 32.	'Flow' and 'circulation.' Stokes' theorem . . . . .	33
33.	Constancy of circulation in a moving circuit . . . . .	35
34, 35.	Irrotational motion in simply-connected spaces; single-valued velocity- potential . . . . .	37

ART.		PAGE
36–39.	Incompressible fluids; tubes of flow. $\phi$ cannot be a maximum or minimum. The velocity cannot be a maximum. Mean value of $\phi$ over a spherical surface . . . . .	38
40, 41.	Conditions of determinateness of $\phi$ . . . . .	41
42–46.	Green's theorem; dynamical interpretation; formula for kinetic energy. Kelvin's theorem of minimum energy . . . . .	43
47, 48.	Multiply-connected regions; 'circuits' and 'barriers' . . . . .	49
49–51.	Irrotational motion in multiply-connected spaces; many-valued velocity-potential; cyclic constants . . . . .	50
52.	Case of incompressible fluids. Conditions of determinateness of $\phi$ . . . . .	53
53–55.	Kelvin's extension of Green's theorem; dynamical interpretation; energy of an irrotationally moving liquid in a cyclic space . . . . .	54
56–58.	'Sources' and 'sinks'; double sources. Irrotational motion of a liquid in terms of surface-distributions of sources . . . . .	57

## CHAPTER IV

## MOTION OF A LIQUID IN TWO DIMENSIONS

59.	Lagrange's stream-function . . . . .	62
60, 60 a.	Relations between stream- and velocity-functions. Two-dimensional sources. Electrical analogies . . . . .	63
61.	Kinetic energy . . . . .	66
62.	Connection with the theory of the complex variable . . . . .	66
63, 64.	Simple types of motion, cyclic and acyclic. Image of a source in a circular barrier. Potential of a row of sources . . . . .	68
65, 66.	Inverse relations. Confocal curves. Flow from an open channel . . . . .	72
67.	General formulae; Fourier method . . . . .	75
68.	Motion of a circular cylinder, without circulation; stream-lines . . . . .	76
69.	Motion of a cylinder with circulation; 'lift.' Trochoidal path under a constant force . . . . .	78
70.	Note on more general problems. Transformation methods; Kutta's problem . . . . .	80
71.	Inverse methods. Motion due to the translation of a cylinder; case of an elliptic section. Flow past an oblique lamina; couple due to fluid pressure . . . . .	83
72.	Motion due to a rotating boundary. Rotating prismatic vessels of various sections. Rotating elliptic cylinder in infinite fluid; general case with circulation . . . . .	86
72 a.	Representation of the effect at a distance of a moving cylinder by a double source . . . . .	90
72 b.	Blasius' expressions for the forces on a fixed cylinder surrounded by an irrotationally moving liquid. Applications; Joukowski's theorem; forces due to a simple source . . . . .	91
73.	Free stream-lines. Schwarz' method of conformal transformation . . . . .	94
74–78.	Examples. Two-dimensional form of Borda's mouthpiece; fluid issuing from a rectilinear aperture; coefficient of contraction. Impact of a stream on a lamina, direct and oblique; resistance. Bobileff's problem . . . . .	96
79.	Discontinuous motions . . . . .	105
80.	Flow on a curved stratum . . . . .	108

## Contents

ix

### CHAPTER V

#### IRROTATIONAL MOTION OF A LIQUID: PROBLEMS IN THREE DIMENSIONS

ART.		PAGE
81, 82.	Spherical harmonics. Maxwell's theory of poles . . . . .	110
83.	Laplace's equation in polar co-ordinates . . . . .	112
84, 85.	Zonal harmonics. Hypergeometric series . . . . .	113
86.	Tesseral and sectorial harmonics . . . . .	116
87, 88.	Conjugate property of surface harmonics. Expansions . . . . .	118
89.	Symbolical solutions of Laplace's equation. Definite integral forms . . . . .	119
90, 91.	Hydrodynamical applications. Impulsive pressures over a spherical surface. Prescribed normal velocity. Energy of motion generated . . . . .	120
91 a.	Examples. Collapse of a bubble. Expansion of a cavity due to internal pressure . . . . .	122
92, 93.	Motion of a sphere in an infinite liquid; inertia coefficient. Effect of a concentric rigid boundary . . . . .	123
94-96.	Stokes' stream-function. Formulæ in spherical harmonics. Stream-lines of a sphere. Images of a simple and a double source in a fluid sphere. Forces on the sphere . . . . .	125
97.	Rankine's inverse method . . . . .	130
98, 99.	Motion of two spheres in a liquid. Kinematical formulæ. Inertia coefficients . . . . .	130
100, 101.	Cylindrical harmonics. Solutions of Laplace's equation in terms of Bessel's functions. Expansion of an arbitrary function . . . . .	134
102.	Hydrodynamical examples. Flow through a circular aperture. Inertia coefficient of a circular disk . . . . .	137
103-106.	Ellipsoidal harmonics for an oblate ellipsoid. Translation and rotation of an oblate ellipsoid in a liquid . . . . .	139
107-109.	Harmonics for a planetary ellipsoid. Flow through a circular aperture. Stream-lines of a circular disk. Translation and rotation of a planetary ellipsoid . . . . .	142
110.	Motion of a fluid in an ellipsoidal vessel . . . . .	146
111.	General orthogonal co-ordinates. Transformation of $\nabla^2\phi$ . . . . .	148
112.	General ellipsoidal co-ordinates; confocal quadrics . . . . .	149
113.	Flow through an elliptic aperture . . . . .	150
114, 115.	Translation and rotation of an ellipsoid in liquid; inertia coefficients . . . . .	152
116.	References to other problems . . . . .	156
	APPENDIX: The hydrodynamical equations referred to general orthogonal co-ordinates . . . . .	156

### CHAPTER VI

#### ON THE MOTION OF SOLIDS THROUGH A LIQUID: DYNAMICAL THEORY

117, 118.	Kinematical formulæ for the case of a single body . . . . .	160
119.	Theory of the 'impulse' . . . . .	161
120.	Dynamical equations relative to axes fixed in the body . . . . .	162
121, 121 a.	Kinetic energy; coefficients of inertia. Representation of the fluid motion at a distance by a double source . . . . .	163
122, 123.	Components of impulse. Reciprocal formulæ . . . . .	166

ART.		PAGE
124.	Expressions for the hydrodynamic forces. The three permanent translations; stability . . . . .	168
125.	The possible modes of steady motion. Motion due to an impulsive couple . . . . .	170
126.	Types of hydrokinetic symmetry . . . . .	172
127–129.	Motion of a solid of revolution. Stability of motion parallel to the axis. Influence of rotation. Other types of steady motion . . . . .	174
130.	Motion of a ‘helicoid’ . . . . .	179
131.	Inertia coefficients of a fluid contained in a rigid envelope . . . . .	180
132–134.	Case of a perforated solid with cyclic motion through the apertures. Steady motion of a ring; condition for stability . . . . .	180
134 a.	The hydrodynamic forces on a cylinder moving in two dimensions . . . . .	184
135, 136.	Lagrange’s equations of motion in generalized co-ordinates. Hamiltonian principle. Adaptation to hydrodynamics . . . . .	187
137, 138.	Examples. Motion of a sphere near a rigid boundary. Motion of two spheres in the line of centres . . . . .	190
139–141.	Modification of Lagrange’s equations in the case of cyclic motion; ignorance of co-ordinates. Equations of a gyrostatic system . . . . .	192
142, 143.	Kineto-statics. Hydrodynamic forces on a solid immersed in a non-uniform stream . . . . .	197
144.	Note on the intuitive extension of dynamic principles . . . . .	201

## CHAPTER VII

## VORTEX MOTION

145.	‘Vortex-lines’ and ‘vortex-filaments’; kinematical properties . . . . .	202
146.	Persistence of vortices; Kelvin’s proof. Equations of Cauchy, Stokes, and Helmholtz. Motion in a fixed ellipsoidal envelope, with uniform vorticity . . . . .	203
147.	Conditions of determinateness . . . . .	207
148, 149.	Velocity in terms of expansion and vorticity; electromagnetic analogy. Velocities due to an isolated vortex . . . . .	208
150.	Velocity-potential due to a vortex . . . . .	211
151.	Vortex-sheets . . . . .	212
152, 153.	Impulse and energy of a vortex-system . . . . .	214
154, 155.	Rectilinear vortices. Stream-lines of a vortex-pair. Other examples . . . . .	219
156.	Investigation of the stability of a row of vortices, and of a double row. Kármán’s ‘vortex-street’ . . . . .	224
157.	Kirchhoff’s theorems on systems of parallel vortices . . . . .	229
158, 159.	Stability of a columnar vortex of finite section; Kirchhoff’s elliptic vortex . . . . .	230
159 a.	Motion of a solid in a liquid of uniform vorticity . . . . .	233
160.	Vortices in a curved stratum of fluid . . . . .	236
161–163.	Circular vortices; potential- and stream-function of an isolated circular vortex; stream-lines. Impulse and energy. Velocity of translation of a vortex-ring . . . . .	236
164.	Mutual influence of vortex-rings. Image of a vortex-ring in a sphere . . . . .	242
165.	General conditions for steady motion of a fluid. Cylindrical and spherical vortices . . . . .	243
166.	References . . . . .	246
166 a.	Bjerknes’ theorems . . . . .	247
167.	Clebsch’s transformation of the hydrodynamical equations . . . . .	248



## Contents

xi

### CHAPTER VIII

#### TIDAL WAVES

ART.		PAGE
168.	General theory of small oscillations; normal modes; forced oscillations .	250
169–174.	Free waves in uniform canal; effect of initial conditions; measuring of the approximations; energy . . . . .	254
175.	Artifice of steady motion . . . . .	261
176.	Superposition of wave-systems; reflection . . . . .	262
177–179.	Effect of disturbing forces; free and forced oscillations in a finite canal .	263
180–184.	Canal theory of the tides. Disturbing potentials. Tides in an equatorial canal, and in a canal parallel to the equator; semi-diurnal and diurnal tides. Canal coincident with a meridian; change of mean level; fortnightly tide. Equatorial canal of finite length; lag of the tide . . . . .	267
185, 186.	Waves in a canal of variable section. Examples of free and forced oscillations; exaggeration of tides in shallow seas and estuaries .	273
187, 188.	Waves of finite amplitude; change of type in a progressive wave. Tides of the second order . . . . .	278
189, 190.	Wave motion in two horizontal dimensions; general equations. Oscillations of a rectangular basin . . . . .	282
191, 192.	Oscillations of a circular basin; Bessel's functions; contour lines. Elliptic basin; approximation to slowest mode . . . . .	284
193.	Case of variable depth. Circular basin . . . . .	291
194–197.	Propagation of disturbances from a centre; Bessel's function of the second kind. Waves due to a local periodic pressure. General formula for diverging waves. Examples of a transient local disturbance . .	293
198–201.	Oscillations of a spherical sheet of water; free and forced waves. Effect of the mutual gravitation of the water. Reference to the case of a sea bounded by meridians and parallels . . . . .	301
202, 203.	Equations of motion of a dynamical system referred to rotating axes .	307
204–205 a.	Small oscillations of a rotating system; stability 'ordinary' and 'secular.' Effect of a <i>small</i> degree of rotation on types and frequencies of normal modes . . . . .	309
205 b.	Approximate calculation of frequencies . . . . .	313
206.	Forced oscillations . . . . .	316
207, 208.	Hydrodynamical examples; tidal oscillations of a rotating plane sheet of water; waves in a straight canal . . . . .	317
209–211.	Rotating circular basin of uniform depth; free and forced oscillations .	320
212.	Circular basin of variable depth . . . . .	326
212 a.	Examples of approximate procedure . . . . .	328
213, 214.	Tidal oscillations on a rotating globe. Laplace's kinetic theory . .	330
215–217.	Symmetrical oscillations. Tides of long period . . . . .	333
218–221.	Diurnal and semi-diurnal tides. Discussion of Laplace's solution . .	340
222, 223.	Hough's investigations; extracts and results . . . . .	347
223 a.	References to further researches . . . . .	352
224.	Modifications of the kinetic theory due to the actual configuration of the ocean; question of phase . . . . .	353
225, 226.	Stability of the ocean. Remarks on the general theory of kinetic stability .	355
	APPENDIX: On Tide-generating Forces . . . . .	358

## CHAPTER IX

## SURFACE WAVES

ART.		PAGE
227.	The two-dimensional problem ; surface conditions . . . . .	363
228.	Standing waves ; lines of motion . . . . .	364
229, 230.	Progressive waves ; orbits of particles. Wave-velocity ; numerical tables. Energy of a simple-harmonic wave-train . . . . .	366
231.	Oscillations of superposed fluids . . . . .	370
232.	Instability of the boundary of two currents . . . . .	373
233, 234.	Artifice of steady motion . . . . .	375
235.	Waves in a heterogeneous liquid . . . . .	378
236, 237.	Group-velocity. Transmission of energy . . . . .	380
238–240.	The Cauchy-Poisson wave-problem ; waves due to an initial local elevation, or to a local impulse. . . . .	384
241.	Kelvin's approximate formula for the effect of a local disturbance in a linear medium. Graphical constructions . . . . .	395
242–246.	Surface-disturbance of a stream. Case of finite depth. Effect of inequalities in its bed . . . . .	398
247.	Waves due to a submerged cylinder . . . . .	410
248, 249.	General theory of waves due to a travelling disturbance. Wave-resistance . . . . .	413
250.	Waves of finite height ; waves of permanent type. Limiting form . . . . .	417
251.	Gerstner's rotational waves . . . . .	421
252, 253.	Solitary waves. Oscillatory waves of Korteweg and De Vries . . . . .	423
254.	Helmholtz' dynamical condition for waves of permanent type . . . . .	427
255, 256.	Wave-propagation in two horizontal dimensions. Effect of a local disturbance. Effect of a travelling pressure-disturbance ; wave-patterns . . . . .	429
256 a, 256 b.	Travelling disturbances of other types. Ship-waves. Wave-resistance. Effect of finite depth on the wave-pattern . . . . .	437
257–259.	Standing waves in limited masses of water. Transverse oscillation in canals of triangular, and semi-circular section . . . . .	440
260, 261.	Longitudinal oscillations ; canal of triangular section ; edge-waves . . . . .	445
262–264.	Oscillations of a liquid globe, lines of motion. Ocean of uniform depth on a spherical nucleus . . . . .	450
265.	Capillarity. Surface-condition . . . . .	455
266.	Capillary waves. Group-velocity . . . . .	456
267, 268.	Waves under gravity and capillarity. Minimum wave-velocity. Waves on the boundary of two currents . . . . .	458
269.	Waves due to a local disturbance. Effect of a travelling disturbance ; waves and ripples . . . . .	462
270–272.	Surface-disturbance of a stream ; formal investigation. Fish-line problem. Wave-patterns . . . . .	464
273, 274	Vibrations of a cylindrical column of liquid. Instability of a jet . . . . .	471
275	Oscillations of a liquid globe, and of a bubble . . . . .	473

*Contents*

xiii

## CHAPTER X

## WAVES OF EXPANSION

ART.		PAGE
276–280.	Plane waves; velocity of sound; energy of a wave-system . . . .	476
281–284.	Plane waves of finite amplitude; methods of Riemann and Earnshaw. Condition for permanence of type; Rankine's investigations. Waves of approximate discontinuity . . . . .	481
285, 286.	Spherical waves. Solution in terms of initial conditions . . . . .	489
287, 288.	General equation of sound-waves. Equation of energy. Determinateness of solutions . . . . .	492
289.	Simple-harmonic vibrations. Simple and double sources. Emission of energy . . . . .	496
290.	Helmholtz' adaptation of Green's theorem. Velocity-potential in terms of surface-distributions of sources. Kirchhoff's formula . . . . .	498
291.	Periodic disturbing forces . . . . .	501
292.	Applications of spherical harmonics. General formulae . . . . .	503
293.	Vibrations of air in a spherical vessel. Vibrations of a spherical stratum	506
294.	Propagation of waves outwards from a spherical surface; attenuation due to lateral motion . . . . .	508
295.	Influence of the air on the oscillations of a ball-pendulum; correction for inertia; damping . . . . .	510
296–298.	Scattering of sound-waves by a spherical obstacle. Impact of waves on a movable sphere; case of synchronism . . . . .	511
299, 300.	Diffraction when the wave-length is relatively large: by a flat disk, by an aperture in a plane screen, and by an obstacle of any form . . . . .	517
301.	Solution of the equation of sound in spherical harmonics. Conditions at a wave-front . . . . .	521
302.	Sound-waves in two dimensions. Effect of a transient source; comparison with the one- and three-dimensional cases . . . . .	524
303, 304.	Simple-harmonic vibrations; solutions in Bessel functions. Oscillating cylinder. Scattering of waves by a cylindrical obstacle . . . . .	527
305.	Approximate theory of diffraction of long waves in two dimensions. Diffraction by a flat blade, and by an aperture in a thin screen . . . . .	531
306, 307.	Reflection and transmission of sound-waves by a grating . . . . .	533
308.	Diffraction by a semi-infinite screen . . . . .	538
309, 310.	Waves propagated vertically in the atmosphere; 'isothermal' and 'con- vective' hypotheses . . . . .	541
311, 311a, 312.	Theory of long atmospheric waves . . . . .	547
313.	General equations of vibration of a gas under constant forces. . . . .	554
314, 315.	Oscillations of an atmosphere on a non-rotating globe . . . . .	556
316.	Atmosphere tides on a rotating globe. Possibility of resonance . . . . .	558

## CHAPTER XI

## VISCOSITY

ART.		PAGE
317, 318.	Theory of dissipative forces. One degree of freedom; free and forced oscillations. Effect of friction on phase . . . . .	562
319.	Application to tides in equatorial canal; tidal lag and tidal friction . . . . .	565
320.	Equations of dissipative systems in general; frictional and gyrostatic terms. Dissipation function . . . . .	567
321.	Oscillations of a dissipative system about a configuration of absolute equilibrium . . . . .	568
322.	Effect of gyrostatic terms. Example of two degrees of freedom; disturbing forces of long period . . . . .	570
323–325.	Viscosity of fluids; specification of stress; formulae of transformation . . . . .	571
326, 327.	The stresses as linear functions of rates of strain. Coefficient of viscosity. Boundary-conditions; question of slipping . . . . .	574
328.	Dynamical equations. The modified Helmholtz equations; diffusion of vorticity . . . . .	576
329.	Dissipation of energy by viscosity . . . . .	579
330, 330 a.	Flow of a liquid between parallel planes. Hele Shaw's experiments. Theory of lubrication; example . . . . .	581
331, 332.	Flow through a pipe of circular section; Poiseuille's laws; question of slipping. Other forms of section . . . . .	585
333, 334.	Cases of steady rotation. Practical limitations . . . . .	587
334 a.	Examples of variable motion. Diffusion of a vortex. Effect of surface-forces on deep water . . . . .	590
335, 336.	Slow steady motion; general solution in spherical harmonics; formulae for the stresses . . . . .	594
337.	Rectilinear motion of a sphere; resistance; terminal velocity; stream-lines. Case of a liquid sphere; and of a solid sphere, with slipping . . . . .	597
338.	Method of Stokes; solutions in terms of the stream-function . . . . .	602
339.	Steady motion of an ellipsoid . . . . .	604
340, 341.	Steady motion in a constant field of force . . . . .	605
342.	Steady motion of a sphere; Oseen's criticism, and solution . . . . .	608
343, 343 a.	Steady motion of a cylinder, treated by Oseen's method. References to other investigations . . . . .	614
344.	Dissipation of energy in steady motion; theorems of Helmholtz and Korteweg. Rayleigh's extension . . . . .	617
345–347.	Problems of periodic motion. Laminar motion, diffusion of vorticity. Oscillating plane. Periodic tidal force; feeble influence of viscosity in rapid motions . . . . .	619
348–351.	Effect of viscosity on water-waves. Generation of waves by wind. Calming effect of oil on waves . . . . .	623
352, 353.	Periodic motion with a spherical boundary; general solution in spherical harmonics . . . . .	632
354.	Applications; decay of motion in a spherical vessel; torsional oscillations of a hollow sphere containing liquid . . . . .	637
355.	Effect of viscosity on the oscillations of a liquid globe . . . . .	639
356.	Effect on the rotational oscillations of a sphere, and on the vibrations of a pendulum . . . . .	641
357.	Notes on two-dimensional problems . . . . .	644

## Contents

XV

ART.	PAGE
358. Viscosity in gases; dissipation function . . . . .	645
359, 360. Damping of plane waves of sound by viscosity; combined effect of viscosity and thermal conduction . . . . .	646
360 a. Waves of permanent type, as affected by viscosity alone . . . . .	650
360 b. Absorption of sound by porous bodies . . . . .	652
361. Effect of viscosity on diverging waves . . . . .	654
362, 363. Effect on the scattering of waves by a spherical obstacle, fixed or free . . . . .	657
364. Damping of sound-waves in a spherical vessel . . . . .	661
365, 366. Turbulent motion. Reynolds' experiments; critical velocities of water in a pipe; law of resistance. Inferences from theory of dimensions . . . . .	663
366 a. Motion between rotating cylinders . . . . .	667
366 b. Coefficient of turbulence; 'eddy' or 'molar' viscosity . . . . .	668
366 c. Turbulence in the atmosphere; variation of wind with height . . . . .	669
367, 368. Theoretical investigations of Rayleigh and Kelvin . . . . .	670
369. Statistical method of Reynolds . . . . .	674
370. Resistance of fluids. Criticism of the discontinuous solutions of Kirchhoff and Rayleigh . . . . .	678
370 a. Kármán's formula for resistance . . . . .	680
370 b. Lift due to circulation . . . . .	681
371. Dimensional formulae. Relations between model and full-scale . . . . .	682
371 a, b, c. The boundary layer. Note on the theory of the aerofoil . . . . .	684
371 d, e, f, g. Influence of compressibility. Failure of stream-line flow at high speeds . . . . .	691

## CHAPTER XII

### ROTATING MASSES OF LIQUID

372. Forms of relative equilibrium. General theorems . . . . .	697
373. Formulae relating to attraction of ellipsoids. Potential energy of an ellipsoidal mass . . . . .	700
374. Maclaurin's ellipsoids. Relations between eccentricity, angular velocity and angular momentum; numerical tables . . . . .	701
375. Jacobi's ellipsoids. Linear series of ellipsoidal forms of equilibrium. Numerical results . . . . .	704
376. Other special forms of relative equilibrium. Rotating annulus . . . . .	707
377. General problem of relative equilibrium; Poincaré's investigation. Linear series of equilibrium forms; limiting forms and forms of bifurcation. Exchange of stabilities . . . . .	710
378–380. Application to a rotating system. Secular stability of Maclaurin's and Jacobi's ellipsoids. The pear-shaped figure of equilibrium . . . . .	713
381. Small oscillations of a rotating ellipsoidal mass; Poincaré's method. References . . . . .	717
382. Dirichlet's investigations; references. Finite gravitational oscillations of a liquid ellipsoid without rotation. Oscillations of a rotating ellipsoid of revolution . . . . .	719
383. Dedekind's ellipsoid. The irrotational ellipsoid. Rotating elliptic cylinder . . . . .	721
384. Free and forced oscillations of a rotating ellipsoidal shell containing liquid. Precession . . . . .	724
385. Precession of a liquid ellipsoid . . . . .	728
LIST OF AUTHORS CITED . . . . .	731
INDEX . . . . .	734

## FOREWORD

The publication of a paperback edition of Lamb's *Hydrodynamics* by Cambridge University Press is a remarkable scientific event, attesting to the enduring vitality of a classic text. First published in 1879 in a much smaller version entitled *A Treatise on the Motion of Fluids*, this book was revised in 1895 under its present title. Successive revisions in 1906, 1916 and 1924 led to the final sixth edition in 1932, which is now being reprinted. Even after 60 years, Lamb's book is owned and used by most fluid dynamicists and is occasionally employed as a text in fluid dynamics courses.

A discussion of the historical context of *Hydrodynamics*, in particular the state of fluid mechanics during Lamb's life, will partly explain the reasons for this longevity. An understanding of the development of the subject may also ease some of the difficulties that the modern reader might have with Lamb's notation and scientific viewpoint. Finally this introduction will hopefully help to document the unique contemporary value of *Hydrodynamics*.

The subject of hydrostatics was founded in the third century B.C. by Archimedes in his book *On Floating Bodies*. Although there were some interesting and well-known observations of fluid motion by Leonardo da Vinci in the fifteenth century, the initial scientific investigation of fluid motion was performed by Sir Isaac Newton in *Principia* (1687) in which he considered the resistance to an object moving through air or liquid and the motion of water waves. The first coherent account of the subject however was that of Daniel Bernoulli whose book *Hydrodynamics* (1738) contained "Bernoulli's law" relating pressure and velocity in an incompressible fluid, as well as a number of its consequences. Leonhard Euler then derived the equations of continuity and momentum for a frictionless fluid in 1755. He derived the equations for both a compressible and an incompressible fluid, and he expressed the equations in a fixed "Eulerian" coordinate system, as well as in a "Lagrangian" coordinate system that moves with the fluid. J.L. Lagrange later took up the subject, without crediting Euler. In particular, along with Laplace and Cauchy, he developed the theory of velocity fields generated by a potential.

The stress tensor for a viscous fluid and the resulting Navier–Stokes equations were first derived by Claude L.M.H. Navier in 1821 and by S.D. Poisson in 1829. This ended a first period in the development of hydrodynamics during which the basic flow equations and their properties were derived, stimulating much of the development of the theory of partial differential equations, but with little progress on solution of fluid flow problems.

The subsequent period from approximately 1840 to 1920 produced many



outstanding analytical successes, which are beautifully captured in Lamb's book. In 1845 George Gabriel Stokes rederived Navier's results from basic mechanical principles and formulated the no-slip boundary condition, which was controversial for many years. The form of the stress tensor received additional confirmation through Maxwell's derivation of the coefficients of viscosity and heat conductivity from kinetic theory in 1866. Stokes later solved the problem of a spherical particle moving through a viscous liquid in which inertia is negligible (§337; this and the numbers below refer to section in *Hydrodynamics*). He obtained the force law  $P = 6\pi\mu aU$  for the force  $P$  on particle of radius  $a$  moving at speed  $U$  in a fluid of viscosity  $\mu$ . This theory was modified by Oseen in 1910 to include the effects of inertia (§342). Osborne Reynolds also applied Stokes's theory to derive a theory of lubrication (§330, 330a).

At the same time great progress was made at the other extreme for an incompressible, inviscid fluid. Stokes again led the way. He solved the problem of a sphere moving through an ideal fluid (§92) and found the added mass to be equal to half the mass of the displaced fluid. This solution does not include a wake, however, since it is symmetric from front to back. A theory of two-dimensional flows bounded by solid walls and free stream lines of constant pressure (§73) was initiated by Helmholtz and developed by Kirchhoff. One result of this type is flow past a flat plate with free streamlines emanating from the ends of the plate (§76-77), which was first derived by Kirchhoff in 1869 and more fully discussed by Lord Rayleigh in 1876. The region of no flow between the two free streamlines can be interpreted as a wake, but it does not provide agreement with experimental results.

Equally successful in this period was the development of the theory of water waves and tidal waves. Although waves in deep water were first examined by Cauchy and Poisson early in the nineteenth century, the real treasures of the subject were discovered later. In an 1876 examination question, Stokes gave the first analytic explanation of the observed dispersion of water waves (§236, 237). He introduced the concept of group velocity, which was generalized by Rayleigh in his research papers and in his book *Theory of Sound* (1877) and then further developed by Osborne Reynolds. Lord Kelvin (William Thomson) introduced the method of stationary phase to describe the interference patterns of water waves. He applied these results to waves produced by ships and derived the fascinating result that the wake of a ship has an angle of  $19.3^\circ$ , independent of its speed (§256). The theory of nonlinear waves was initiated by Stokes in 1847, who showed that wave speed depends on the amplitude (§250). The particular case of solitary waves, first observed by Scott Russell in 1844, was analyzed by Boussinesq and Rayleigh (§252, 253). Further investigation of this problem by Korteweg and de Vries in 1895 is referred to in a footnote to §253; their "KdV equation" was shown to be completely integrable in the 1960's and was the starting point for a new mathematical subject.

Vortex motion was another topic that was developed during the nineteenth

century, and it provided much of the original motivation for Lamb's *Treatise*. The study of vortex motion was started by Hermann von Helmholtz in his paper of 1858 and was further developed and simplified by Kelvin in a paper of 1869. The persistence of vorticity, now called Kelvin's theorem (§33), and Helmholtz's simple equations for the evolution of the vorticity (§146) provide a general method with which to analyze unsteady flows. Special solutions, such as vortex sheets, rectilinear vortices (i.e. two dimensional point vortices), elliptical vortex patches, vortex streets and vortex rings, provide a wealth of examples for understanding real fluid flows. Kelvin even proposed a theory of "vortex atoms" based on vortex rings in the ether, which has long since been discarded.

This was the scientific environment in which Lamb began his career. Born in 1849 to a father who was a foreman in a cotton mill, Horace Lamb received his B.A. degree in 1872 from Trinity College, where Stokes and Maxwell were among his teachers. He stayed on at Trinity as a Fellow and Lecturer for three years, during which time he first gave the lectures that were the basis for his monograph *A Treatise on the Motion of Fluids*. His was one of the first courses to include the new theories of water waves, free surfaces using complex variables, and vortex motion. Students who attended these lectures and reviewers of Lamb's text were enthusiastic about the striking depth and originality of his exposition.

In 1875 Lamb left his position at Trinity, where Fellows were still under a rule of celibacy, and after marrying he accepted a position as Professor at the recently established University of Adelaide. He remained in Australia until 1885 when he returned to Manchester as Professor of Pure Mathematics in Owens College. Except for a change of his title to Professor of Mathematics (Pure and Applied), this was his position until his retirement in 1920. At that time he returned to Cambridge as an Honorary Fellow at Trinity and was appointed to an honorary lectureship, the Rayleigh Lectureship, in the Mathematics Institute at the University. Lamb was elected a Fellow of the Royal Society in 1884 and received its Royal and Copley Medals. In addition to many other honors, he was knighted in 1931. He was scientifically active throughout his retirement until his death in 1934.

Lamb's research contributions were primarily on wave motion and vibrations, particularly on spherical bodies. His papers on oscillatory modes of an elastic sphere and on the propagation of surface waves on a sphere were seminal works in theoretical seismology and earthquake wave transmission. One of the wave types predicted by Lamb's theory of 1882 was only first observed in a Chilean earthquake of 1960. Lamb made equally fundamental contributions to the theory of tides and terrestrial magnetism. The first satisfactory explanation of the marked difference between tides observed in different parts of the oceans was due to Lamb, and he calculated the deflection of the earth's surface caused



by tidal loading. He also gave an analysis of the diurnal variation of the earth's magnetism.

*Hydrodynamics* contains numerous original contributions of Lamb, which are not always easy to detect because of the author's modesty. Examples include the oscillations of a viscous sphere (§355), the phase of the tides (§184), and a simplified derivation of Oseen's theory for motion of a sphere in a viscous liquid (§342), as well as its application to the motion of a cylinder (§343). Moreover much of the exposition of Chapter VIII on tidal waves and that on motion of a solid through a liquid in Chapter VI, in particular for a perforated solid (§132-134), was due to Lamb.

In addition to *Hydrodynamics*, Lamb wrote a number of other textbooks which were widely used at the time, including *Infinitesimal Calculus* (1897), *The Dynamical Theory of Sound* (1910), *Statics* (1912), *Dynamics* (1914) and *Higher Mechanics* (1920). *Hydrodynamics* itself was extremely well received and influential. For example, Rayleigh wrote an enthusiastic review of the fourth edition in 1916, describing Lamb's text as a vast improvement over earlier texts, which he described as "arid in the extreme." He further stated that "to almost all parts of his subject he has made entirely original contributions," and "on all of these subjects the reader will find expositions which could scarcely be improved."

A new period in the development of hydrodynamics started around the turn of the century. The fluid flow phenomena and solutions that were developed in the nineteenth century formed an impressive analytic theory, which occupies the heart of Lamb's text. This theory demonstrated the power of mathematical technique combined with physical reasoning, but it was not yet of much practical value. In his 1916 review of *Hydrodynamics*, Rayleigh concluded with a call for more coordination between theory and experimental results, stating that "one can scarcely deny that much of [theoretical hydrodynamics] is out of touch with reality." Indeed, at that time there was little agreement between theory and experiment for many flows, notably for the motion of an obstacle through air or liquid (except in the case of slow flow for which inertia is negligible). Neither Stokes' irrotational solutions (§93) nor the solutions with free streamlines (§76) developed by Kirchoff give correct results, as pointed out in §370. Moreover in the wake of an obstacle such as a solid cylinder, double trails of vortices of alternating sign were observed as early as 1902 by Ahlborn (§370a) and were analyzed by von Karman in 1911 (§156).

An understanding of the role of viscosity and vorticity in flow around an obstacle was not possible until the development of boundary layer theory by Ludwig Prandtl beginning in 1904 (§371a). Around the same time the lift on an airfoil was explained by Lanchester's lifting line theory (§370b) and the general theorems of Blasius (§72b). Generation of a wake however requires detachment of the boundary layer as a mechanism for injecting vorticity into the outer flow, as briefly described in §371b. Lamb's discussion of boundary

*Foreword*

xxi

layers and the role of vorticity is incomplete; for example, the important result that vorticity in a fluid flow is generated only at boundaries receives only a brief mention in §328. In addition the statement in §371b that separation of the boundary layer for an impulsively moving cylinder first occurs at  $180^\circ$  has recently been disproved by van Dommelen and Shen through numerical computation. Indeed the subject of boundary layers and separation is still an area of intense research, mainly through experiment, triple-deck theory and direct numerical simulation. For example, there is still considerable debate over the form of the steady solution for flow past a sphere at high Reynolds number, although this flow is unstable and almost certainly not physically observable.

An even more problematic flow phenomena was first observed by O. Reynolds in 1883 (§365, 366). His experiments with pipe flow showed agreement with Poiseuille's theory below a critical value of the Reynolds number (Reynolds number is defined in a footnote in §366), but above that value the flow becomes turbulent. Lamb credits Kelvin with coining the name "turbulence" for these troublesome flows. Similar results for rotating flows were first observed by G.I. Taylor in 1922. Reynolds also developed the idea of an eddy viscosity (§366b) to describe the macroscopic behavior of a turbulent flow.

Following these early investigations, outstanding progress has been made in experimental technique, with results such as the recent discovery of spatial coherence in developing turbulent flows. There have been some equally significant theoretical advances, particularly A. Kolmogorov's 1941 theory of energy cascade and the inertial range, although corrections are believed necessary to accurately account for intermittency. The mathematical theory of chaos is apparently inadequate for describing the many degrees of freedom present in a turbulent flow. Nevertheless it provides an effective description for many flows that are complicated, but less than turbulent, such as those seen during the development of oscillating patterns in rotating flow. Finally numerical simulations of turbulent flows are proving to be quite valuable, although they are severely limited by both computational speed and memory size.

The theory of shock waves in gases was another topic under development during the latter part of Lamb's life, and a treatment of the subject is initiated in §284 of Chapter X on Waves of Expansion. Rankine's derivation (1870) of the jump conditions for mass and momentum conservation across a shock is presented; then Lamb repeats Rayleigh's objection that the energy cannot be conserved across such a jump. The mistake in Rayleigh's argument was in his implicit assumption that the entropy is constant across a shock. Indeed, Hugoniot's correct proposal (1889) that a jump in entropy across the interface changes the equation of state is mentioned by Lamb in a footnote but regarded as physically suspect. This section of *Hydrodynamics* also contains a brief account of the effect of dissipation on the shock profile. The theory of inviscid, as well as viscous, shock waves has since been more completely developed.

Computational Fluid Dynamics (CFD) is almost entirely missing from *Hy-*

*drodynamics.* Before the advent of the digital computer, hand computation of fluid flow problems was performed only with considerable effort. Starting with the wartime work of the 1940's, CFD has been tremendously successful in simulating flows such as shock waves, flow around airfoils and convective flows. Although the most complicated flows, particularly those that are fully three-dimensional, are currently beyond the reach of computation, numerical simulation has emerged as a complementary approach to physical experimentation and analytic theory. Certainly computation entails some new difficulties, including artificial boundary conditions and numerical instabilities, that are not present in physical experiments. Computational experiments do, however, offer decided advantages, such as more freedom in choice of parameter values, better control over noise, and more complete data, particularly for vorticity.

Numerical computation does introduce new pathologies and instabilities that are not well addressed by physical intuition. A result of this has been the revitalization of mathematical theory for fluid dynamics. In his 1916 review, Rayleigh complained that rigorous mathematical analyses of physical problems often “tell us only what we knew before.” This is not the case for numerical problems, however, and mathematical analysis has been instrumental in the development of effective methods for simulating shock waves, for example. A mathematical theory of fluid flows is currently far from complete. Among other issues, there is now considerable debate over the possible development of singularities in three-dimensional inviscid, turbulent flows. For turbulent flows, the mathematical theory is still in its infancy.

In §371b Lamb states the question, originally raised by Oseen, of the inviscid limit ( $\nu \rightarrow 0$ ) for flow around an obstacle such as a sphere, and points out that the limit may be different from the inviscid flow. The solution to this problem is still unknown, in spite of considerable effort both analytically and numerically, and it constitutes one of the major outstanding questions of theoretical fluid dynamics. For example the energy dissipation in a turbulent flow is believed to remain nonzero in the limit of zero viscosity.

The modern reader will notice many differences in style as well as content between Lamb's *Hydrodynamics* and current textbooks. Most noticeable is that matrix-vector notation is absent, which can be quite burdensome for unsuspecting students. Consider for example the elegant matrix-vector formulation of Helmholtz's result

$$\vec{\omega}/\rho = (\partial\vec{x}/\partial\vec{x}_0)\vec{\omega}_0/\rho_0$$

for the vorticity  $\vec{\omega}$  and density  $\rho$  at time  $t$ , in terms of the initial vorticity  $\vec{\omega}_0$  and density  $\rho_0$ , and the derivative of the flow map  $\vec{x}(\vec{x}_0, t)$ . In *Hydrodynamics* it is written out component by component in much less transparent form as equation (3) in §146. In the same section Helmholtz's evolution equation for vorticity

$$D(\vec{\omega}/\rho)/Dt = (\vec{\omega}/\rho) \cdot \nabla\vec{u}$$

is again written out in less transparent form as equation (4) of §146.

Since matrix-vector notation is now so prevalent, one may be surprised to discover that it was not so in Lamb's day. In fact the development of vector analysis has a complicated history. The first attempt to develop an algebra of three-dimensional points was through the use of quaternions, which were discovered by Hamilton in 1843. He considered them to be a three-dimensional analogue of the representation of two dimensional points by complex numbers. In his theory Hamilton introduced the gradient operator  $\nabla$ , which he called "nabla." He urged physicists to adopt quaternions, and James Clerk Maxwell was greatly impressed by Hamilton's theory. In *A Treatise on Electricity and Magnetism*, Maxwell was the first to employ vectors and scalars, interpreting them as the components of the quaternion representation. Maxwell also distinguished the divergence, curl and Laplacian operators.

The initiation of vector analysis as a distinct subject was made independently by Josiah Willard Gibbs and Oliver Heaviside, and was popularized through *Vector Analysis* by Gibbs and E.B. Wilson (1901) and *Electromagnetic Theory* by Heaviside (1893). They developed the algebra and geometry of vectors, and they defined the scalar and vector products for three-dimensional vectors. Moreover, Heaviside was the first to write Maxwell's equations in the elegant vector form that is now familiar; Maxwell had always written them out component by component, just as Lamb does for the equations of hydrodynamics.

Engineers and physicists were quick to follow the lead of Gibbs, who was a physical chemist, and Heaviside, who was an electrical engineer, since they found quaternions to be cumbersome and too far removed from the geometry of Cartesian coordinates. Mathematicians however fiercely resisted vector analysis in favor of quaternions for some time. Finally, vector methods were adopted in analytic and differential geometry, and quaternions faded from the mainstream of mathematics.

The history of matrices is more subtle, since determinants were used long before matrices themselves were studied. In the early 1700's, Maclaurin distinguished the determinant in his study of solutions of simultaneous linear equations. The theory of determinants was further developed in the nineteenth century by Sylvester, who first used the term "matrix" in his studies. A separate study of matrices was finally initiated by Arthur Cayley in his investigations of invariants under linear transformations. He published a fully developed theory, defining matrix multiplication, inversion and transposition, as well as the characteristic equation for a matrix, in "A Memoir on the Theory of Matrices" (1858).

The first exclusive use of matrix-vector notation in a hydrodynamics text was in *Theoretical Hydrodynamics* by L.M. Milne-Thomson, first published in 1938. After describing this as a radical departure from the traditional presenta-

tion in his preface, Milne-Thomson devoted a chapter to the elements of vectors and tensors. In more recent texts, such as G.K. Batchelor's book *An Introduction to Fluid Dynamics*, vectors and matrices are used without comment, since they are now a standard part of an undergraduate education.

Another aspect of *Hydrodynamics* that may trouble modern readers is Lamb's emphasis on exact solutions. This is also one of the main strengths of the text, however, and is a major reason for its lasting popularity. Since Lamb's time, the scarcity of simple exact solutions and the limitations of infinite series expansions have become more apparent. Emphasis is now placed on interpretation of the most important exact solutions and the physical phenomena that they manifest. Techniques of asymptotic analysis have also greatly improved, so that now many more flow problems can be solved through perturbation expansions. Most recently numerical computations have become an extremely effective method for investigating fluid flows, and their role is almost certainly going to increase in the future. Nevertheless, theoretical fluid dynamics is still largely a collection of flow examples and *Hydrodynamics* contains a wealth of them.

Two misprints in the text should be pointed out. In §697 the  $\bar{z}$  component of the rotation vector should be 0 rather than  $-\omega^2 z$ . Another is in the footnote of §17 stating that the preface gives an explanation for the minus sign in the definition  $\bar{u} = -\nabla\phi$  for the potential. Lamb's explanation, which was included in the fifth edition but omitted in the sixth, is that with this choice  $\phi$  is the impulsive pressure, or the potential of an impulsive force, that would start the flow  $\bar{u}$  from rest, rather than one that would stop the flow.

The value of Lamb's *Hydrodynamics* today is first as a storehouse of exact solutions for fluid dynamic problems, as stated earlier. In this aspect it is unequaled by modern texts. There are also certain fluid dynamic topics that are still best expressed in Lamb's book. An example is his discussion of the Hamiltonian formulation for fluid dynamic problems. In particular Chapter VI develops the Hamiltonian approach for the dynamics of solid particles in a fluid, treating the mixture as a single system. As Lamb points out at the end of the chapter, this approach has not been validated in all circumstances, and it seems to be fertile ground for further exploration. A related topic that is often omitted in contemporary texts is Clebsch coordinates, described in §167 at the end of Chapter VII.

Less tangible but equally important is the contact that *Hydrodynamics* provides with an earlier era of fluid dynamics. Lamb gave careful attribution to original sources, which is of great help to anyone interested in the history of fluid mechanics. More important to most readers is the perspective conveyed from a crucial period in the development of this subject. Since the final revision of *Hydrodynamics* in the 1930's, great progress has been made in fluid dynamics. Lamb's treatment of nonlinear water waves, shock waves, fluid dynamic stability, boundary layers and turbulence, for example, suggests many

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*Foreword*

xxv

problems, of which a large number have since been solved but many others remain open. Thus *Hydrodynamics* provides us with a valuable measure of the past progress of fluid dynamics and with a compelling challenge for its future.

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