HYDRODYNAMICS

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PREFACE

THIS may be regarded as the sixth edition of a *Treatise on the Mathematical Theory of the Motion of Fluids*, published in 1879. Subsequent editions, largely remodelled and extended, have appeared under the present title.

In this issue no change has been made in the general plan and arrangement, but the work has again been revised throughout, some important omissions have been made good, and much new matter has been introduced.

The subject has in recent years received considerable developments, in the theory of the tides for instance, and in various directions bearing on the problems of aeronautics, and it is interesting to note that the "classical" Hydrodynamics, often referred to with a shade of depreciation, is here found to have a widening field of practical applications. Owing to the elaborate nature of some of these researches it has not always been possible to fit an adequate account of them into the frame of this book, but attempts have occasionally been made to give some indication of the more important results, and of the methods employed.

As in previous editions, pains have been taken to make due acknowledgment of authorities in the footnotes, but it appears necessary to add that the original proofs have often been considerably modified in the text.

I have again to thank the staff of the University Press for much valued assistance during the printing.

HORACE LAMB

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FOREWORD

The publication of a paperback edition of Lamb's *Hydrodynamics* by Cambridge University Press is a remarkable scientific event, attesting to the enduring vitality of a classic text. First published in 1879 in a much smaller version entitled *A Treatise on the Motion of Fluids*, this book was revised in 1895 under its present title. Successive revisions in 1906, 1916 and 1924 led to the final sixth edition in 1932, which is now being reprinted. Even after 60 years, Lamb's book is owned and used by most fluid dynamicists and is occasionally employed as a text in fluid dynamics courses.

A discussion of the historical context of *Hydrodynamics*, in particular the state of fluid mechanics during Lamb's life, will partly explain the reasons for this longevity. An understanding of the development of the subject may also ease some of the difficulties that the modern reader might have with Lamb's notation and scientific viewpoint. Finally this introduction will hopefully help to document the unique contemporary value of *Hydrodynamics*.

The subject of hydrostatics was founded in the third century B.C. by Archimedes in his book *On Floating Bodies*. Although there were some interesting and well-known observations of fluid motion by Leonardo da Vinci in the fifteenth century, the initial scientific investigation of fluid motion was performed by Sir Isaac Newton in *Principia* (1687) in which he considered the resistance to an object moving through air or liquid and the motion of water waves. The first coherent account of the subject however was that of Daniel Bernoulli whose book *Hydrodynamics* (1738) contained "Bernoulli's law" relating pressure and velocity in an incompressible fluid, as well as a number of its consequences. Leonhard Euler then derived the equations of continuity and momentum for a frictionless fluid in 1755. He derived the equations for both a compressible and an incompressible fluid, and he expressed the equations in a fixed "Eulerian" coordinate system, as well as in a "Lagrangian" coordinate system that moves with the fluid. J.L. Lagrange later took up the subject, without crediting Euler. In particular, along with Laplace and Cauchy, he developed the theory of velocity fields generated by a potential.

The stress tensor for a viscous fluid and the resulting Navier–Stokes equations were first derived by Claude L.M.H. Navier in 1821 and by S.D. Poisson in 1829. This ended a first period in the development of hydrodynamics during which the basic flow equations and their properties were derived, stimulating much of the development of the theory of partial differential equations, but with little progress on solution of fluid flow problems.

The subsequent period from approximately 1840 to 1920 produced many outstanding analytical successes, which are beautifully captured in Lamb's book. In 1845 George Gabriel Stokes rederived Navier's results from basic mechanical principles and formulated the no-slip boundary condition, which was controversial for many years. The form of the stress tensor received additional confirmation through Maxwell's derivation of the coefficients of viscosity and heat conductivity from kinetic theory in 1866. Stokes later solved the problem of a spherical particle moving through a viscous liquid in which inertia is negligible (§337; this and the numbers below refer to section in *Hydrodynamics*). He obtained the force law $P = 6\pi\mu aU$ for the force P on particle of radius a moving at speed U in a fluid of viscosity μ. This theory was modified by Oseen in 1910 to include he effects of inertia (§342). Osborne Reynolds also applied Stokes's theory to derive a theory of lubrication (§330, 330a).

At the same time great progress was made at the other extreme for an incompressible, inviscid fluid. Stokes again led the way. He solved the problem of a sphere moving through an ideal fluid (§92) and found the added mass to be equal to half the mass of the displaced fluid. This solution does not include a wake, however, since it is symmetric from front to back. A theory of two-dimensional flows bounded by solid walls and free stream lines of constant pressure (§73) was initiated by Helmholtz and developed by Kirchhoff. One result of this type is flow past a flat plate with free streamlines emanating from the ends of the plate (§76-77), which was first derived by Kirchoff in 1869 and more fully discussed by Lord Rayleigh in 1876. The region of no flow between the two free streamlines can be interpreted as a wake, but it does not provide agreement with experimental results.

Equally successful in this period was the development of the theory of water waves and tidal waves. Although waves in deep water were first examined by Cauchy and Poisson early in the nineteenth century, the real treasures of the subject were discovered later. In an 1876 examination question, Stokes gave the first analytic explanation of the observed dispersion of water waves (§236, 237). He introduced the concept of group velocity, which was generalized by Rayleigh in his research papers and in his book *Theory of Sound* (1877) and then further developed by Osborne Reynolds. Lord Kelvin (William Thomson) introduced the method of stationary phase to describe the interference patterns of water waves. He applied these results to waves produced by ships and derived the fascinating result that the wake of a ship has an angle of 19.3°, independent of its speed (§256). The theory of nonlinear waves was initiated by Stokes in 1847, who showed that wave speed depends on the amplitude (§250). The particular case of solitary waves, first observed by Scott Russell in 1844, was analyzed by Boussinesq and Rayleigh (§252, 253). Further investigation of this problem by Korteweg and de Vries in 1895 is referred to in a footnote to §253; their "KdV equation" was shown to be completely integrable in the 1960's and was the starting point for a new mathematical subject.

Vortex motion was another topic that was developed during the nineteenth century, and it provided much of the original motivation for Lamb's *Treatise.* The study of vortex motion was started by Hermann von Helmholtz in his paper of 1858 and was further developed and simplified by Kelvin in a paper of 1869. The persistence of vorticity, now called Kelvin's theorem (§33), and Helmholtz's simple equations for the evolution of the vorticity (§146) provide a general method with which to analyze unsteady flows. Special solutions, such as vortex sheets, rectilinear vortices (i.e. two dimensional point vortices), elliptical vortex patches, vortex streets and vortex rings, provide a wealth of examples for understanding real fluid flows. Kelvin even proposed a theory of "vortex atoms" based on vortex rings in the ether, which has long since been discarded.

This was the scientific environment in which Lamb began his career. Born in 1849 to a father who was a foreman in a cotton mill, Horace Lamb received his B.A. degree in 1872 from Trinity College, where Stokes and Maxwell were among his teachers. He stayed on at Trinity as a Fellow and Lecturer for three years, during which time he first gave the lectures that were the basis for his monograph *A Treatise on the Motion of Fluids.* His was one of the first courses to include the new theories of water waves, free surfaces using complex variables, and vortex motion. Students who attended these lectures and reviewers of Lamb's text were enthusiastic about the striking depth and originality of his exposition.

In 1875 Lamb left his position at Trinity, where Fellows were still under a rule of celibacy, and after marrying he accepted a position as Professor at the recently established University of Adelaide. He remained in Australia until 1885 when he returned to Manchester as Professor of Pure Mathematics in Owens College. Except for a change of his title to Professor of Mathematics (Pure and Applied), this was his position until his retirement in 1920. At that time he returned to Cambridge as an Honorary Fellow at Trinity and was appointed to an honorary lectureship, the Rayleigh Lectureship, in the Mathematics Institute at the University. Lamb was elected a Fellow of the Royal Society in 1884 and received its Royal and Copley Medals. In addition to many other honors, he was knighted in 1931. He was scientifically active throughout his retirement until his death in 1934.

Lamb's research contributions were primarily on wave motion and vibrations, particularly on spherical bodies. His papers on oscillatory modes of an elastic sphere and on the propagation of surface waves on a sphere were seminal works in theoretical seismology and earthquake wave transmission. One of the wave types predicted by Lamb's theory of 1882 was only first observed in a Chilean earthquake of 1960. Lamb made equally fundamental contributions to the theory of tides and terrestrial magnetism. The first satisfactory explanation of the marked difference between tides observed in different parts of the oceans was due to Lamb, and he calculated the deflection of the earth's surface caused by tidal loading. He also gave an analysis of the diurnal variation of the earth's magnetism.

Hydrodynamics contains numerous original contributions of Lamb, which are not always easy to detect because of the author's modesty. Examples include the oscillations of a viscous sphere (§355), the phase of the tides (§184), and a simplified derivation of Oseen's theory for motion of a sphere in a viscous liquid (§342), as well as its application to the motion of a cylinder (§343). Moreover much of the exposition of Chapter VIII on tidal waves and that on motion of a solid through a liquid in Chapter VI, in particular for a perforated solid (§132-134), was due to Lamb.

In addition to *Hydrodynamics*, Lamb wrote a number of other textbooks which were widely used at the time, including *Infinitesimal Calculus* (1897), *The Dynamical Theory of Sound* (1910), *Statics* (1912), *Dynamics* (1914) and *Higher Mechanics* (1920). *Hydrodynamics* itself was extremely well received and influential. For example, Rayleigh wrote an enthusiastic review of the fourth edition in 1916, describing Lamb's text as a vast improvement over earlier texts, which he described as "arid in the extreme." He further stated that "to almost all parts of his subject he has made entirely original contributions," and "on all of these subjects the reader will find expositions which could scarcely be improved."

A new period in the development of hydrodynamics started around the turn of the century. The fluid flow phenomena and solutions that were developed in the nineteenth century formed an impressive analytic theory, which occupies the heart of Lamb's text. This theory demonstrated the power of mathematical technique combined with physical reasoning, but it was not yet of much practical value. In his 1916 review of *Hydrodynamics*, Rayleigh concluded with a call for more coordination between theory and experimental results, stating that "one can scarcely deny that much of [theoretical hydrodynamics] is out of touch with reality." Indeed, at that time there was little agreement between theory and experiment for many flows, notably for the motion of an obstacle through air or liquid (except in the case of slow flow for which inertia is negligible). Neither Stokes' irrotational solutions (§93) nor the solutions with free streamlines (§76) developed by Kirchoff give correct results, as pointed out in §370. Moreover in the wake of an obstacle such as a solid cylinder, double trails of vortices of alternating sign were observed as early as 1902 by Ahlborn (§370a) and were analyzed by von Karman in 1911 (§156).

An understanding of the role of viscosity and vorticity in flow around an obstacle was not possible until the development of boundary layer theory by Ludwig Prandtl beginning in 1904 (§371a). Around the same time the lift on an airfoil was explained by Lanchester's lifting line theory (§370b) and the general theorems of Blasius (§72b). Generation of a wake however requires detachment of the boundary layer as a mechanism for injecting vorticity into the outer flow, as briefly described in §371b. Lamb's discussion of boundary layers and the role of vorticity is incomplete; for example, the important result that vorticity in a fluid flow is generated only at boundaries receives only a brief mentioned in §328. In addition the statement in §371b that separation of the boundary layer for an impulsively moving cylinder first occurs at 180° has recently been disproved by van Dommelen and Shen through numerical computation. Indeed the subject of boundary layers and separation is still an area of intense research, mainly through experiment, triple-deck theory and direct numerical simulation. For example, there is still considerable debate over the form of the steady solution for flow past a sphere at high Reynolds number, although this flow is unstable and almost certainly not physically observable.

An even more problematic flow phenomena was first observed by O. Reynolds in 1883 (§365, 366). His experiments with pipe flow showed agreement with Poiseuille's theory below a critical value of the Reynolds number (Reynolds number is defined in a footnote in §366), but above that value the flow becomes turbulent. Lamb credits Kelvin with coining the name "turbulence" for these troublesome flows. Similar results for rotating flows were first observed by G.I. Taylor in 1922. Reynolds also developed the idea of an eddy viscosity (§366b) to describe the macroscopic behavior of a turbulent flow.

Following these early investigations, outstanding progress has been made in experimental technique, with results such as the recent discovery of spatial coherence in developing turbulent flows. There have been some equally significant theoretical advances, particularly A. Kolmogorov's 1941 theory of energy cascade and the inertial range, although corrections are believed necessary to accurately account for intermittency. The mathematical theory of chaos is apparently inadequate for describing the many degrees of freedom present in a turbulent flow. Nevertheless it provides an effective description for many flows that are complicated, but less than turbulent, such as those seen during the development of oscillating patterns in rotating flow. Finally numerical simulations of turbulent flows are proving to be quite valuable, although they are severely limited by both computational speed and memory size.

The theory of shock waves in gases was another topic under development during the latter part of Lamb's life, and a treatment of the subject is initiated in §284 of Chapter X on Waves of Expansion. Rankine's derivation (1870) of the jump conditions for mass and momentum conservation across a shock is presented; then Lamb repeats Rayleigh's objection that the energy cannot be conserved across such a jump. The mistake in Rayleigh's argument was in his implicit assumption that the entropy is constant across a shock. Indeed, Hugoniot's correct proposal (1889) that a jump in entropy across the interface changes the equation of state is mentioned by Lamb in a footnote but regarded as physically suspect. This section of *Hydrodynamics* also contains a brief account of the effect of dissipation on the shock profile. The theory of inviscid, as well as viscous, shock waves has since been more completely developed.

Computational Fluid Dynamics (CFD) is almost entirely missing from *Hy-*

drodynamics. Before the advent of the digital computer, hand computation of fluid flow problems was performed only with considerable effort. Starting with the wartime work of the 1940's, CFD has been tremendously successful in simulating flows such as shock waves, flow around airfoils and convective flows. Although the most complicated flows, particularly those that are fully three-dimensional, are currently beyond the reach of computation, numerical simulation has emerged as a complementary approach to physical experimentation and analytic theory. Certainly computation entails some new difficulties, including artificial boundary conditions and numerical instabilities, that are not present in physical experiments. Computational experiments do, however, offer decided advantages, such as more freedom in choice of parameter values, better control over noise, and more complete data, particularly for vorticity.

Numerical computation does introduce new pathologies and instabilities that are not well addressed by physical intuition. A result of this has been the revitalization of mathematical theory for fluid dynamics. In his 1916 review, Rayleigh complained that rigorous mathematical analyses of physical problems often "tell us only what we knew before." This is not the case for numerical problems, however, and mathematical analysis has been instrumental in the development of effective methods for simulating shock waves, for example. A mathematical theory of fluid flows is currently far from complete. Among other issues, there is now considerable debate over the possible development of singularities in three-dimensional inviscid, turbulent flows. For turbulent flows, the mathematical theory is still in its infancy.

In §371b Lamb states the question, originally raised by Oseen, of the inviscid limit ($\nu \to 0$) for flow around an obstacle such as a sphere, and points out that the limit may be different from the inviscid flow. The solution to this problem is still unknown, in spite of considerable effort both analytically and numerically, and it constitutes one of the major outstanding questions of theoretical fluid dynamics. For example the energy dissipation in a turbulent flow is believed to remain nonzero in the limit of zero viscosity.

The modern reader will notice many differences in style as well as content between Lamb's *Hydrodynamics* and current textbooks. Most noticeable is that matrix-vector notation is absent, which can be quite burdensome for unsuspecting students. Consider for example the elegant matrix-vector formulation of Helmholtz's result

$$\vec{\omega}/\rho = (\partial\vec{x}/\partial\vec{x}_0)\vec{\omega}_0/\rho_0$$

for the vorticity $\vec{\omega}$ and density ρ at time t, in terms of the initial vorticity $\vec{\omega}_0$ and density ρ_0, and the derivative of the flow map $\vec{x}(\vec{x}_0, t)$. In *Hydrodynamics* it is written out component by component in much less transparent form as equation (3) in §146. In the same section Helmholtz's evolution equation for vorticity

$$D(\vec{\omega}/\rho)/Dt = (\vec{\omega}/\rho)\cdot\nabla\vec{u}$$

is again written out in less transparent form as equation (4) of §146.

Since matrix-vector notation is now so prevalent, one may be surprised to discover that it was not so in Lamb's day. In fact the development of vector analysis has a complicated history. The first attempt to develop an algebra of three-dimensional points was through the use of quaternions, which were discovered by Hamilton in 1843. He considered them to be a three-dimensional analogue of the representation of two dimensional points by complex numbers. In his theory Hamilton introduced the gradient operator ∇, which he called "nabla." He urged physicists to adopt quaternions, and James Clerk Maxwell was greatly impressed by Hamilton's theory. In *A Treatise on Electricity and Magnetism*, Maxwell was the first to employ vectors and scalars, interpreting them as the components of the quaternion representation. Maxwell also distinguished the divergence, curl and Laplacian operators.

The initiation of vector analysis as a distinct subject was made independently by Josiah Willard Gibbs and Oliver Heaviside, and was popularized through *Vector Analysis* by Gibbs and E.B. Wilson (1901) and *Electromagnetic Theory* by Heaviside (1893). They developed the algebra and geometry of vectors, and they defined the scalar and vector products for three-dimensional vectors. Moreover, Heaviside was the first to write Maxwell's equations in the elegant vector form that is now familiar; Maxwell had always written them out component by component, just as Lamb does for the equations of hydrodynamics.

Engineers and physicists were quick to follow the lead of Gibbs, who was a physical chemist, and Heaviside, who was an electrical engineer, since they found quaternions to be cumbersome and too far removed from the geometry of Cartesian coordinates. Mathematicians however fiercely resisted vector analysis in favor of quaternions for some time. Finally, vector methods were adopted in analytic and differential geometry, and quaternions faded from the mainstream of mathematics.

The history of matrices is more subtle, since determinants were used long before matrices themselves were studied. In the early 1700's, Maclaurin distinguished the determinant in his study of solutions of simultaneous linear equations. The theory of determinants was further developed in the nineteenth century by Sylvester, who first used the term "matrix" in his studies. A separate study of matrices was finally initiated by Arthur Cayley in his investigations of invariants under linear transformations. He published a fully developed theory, defining matrix multiplication, inversion and transposition, as well as the characteristic equation for a matrix, in "A Memoir on the Theory of Matrices" (1858).

The first exclusive use of matrix-vector notation in a hydrodynamics text was in *Theoretical Hydrodynamics* by L.M. Milne-Thomson, first published in 1938. After describing this as a radical departure from the traditional presentation in his preface, Milne-Thomson devoted a chapter to the elements of vectors and tensors. In more recent texts, such as G.K. Batchelor's book *An Introduction to Fluid Dynamics*, vectors and matrices are used without comment, since they are now a standard part of an undergraduate education.

Another aspect of *Hydrodynamics* that may trouble modern readers is Lamb's emphasis on exact solutions. This is also one of the main strengths of the text, however, and is a major reason for its lasting popularity. Since Lamb's time, the scarcity of simple exact solutions and the limitations of infinite series expansions have become more apparent. Emphasis is now placed on interpretation of the most important exact solutions and the physical phenomena that they manifest. Techniques of asymptotic analysis have also greatly improved, so that now many more flow problems can be solved through perturbation expansions. Most recently numerical computations have become an extremely effective method for investigating fluid flows, and their role is almost certainly going to increase in the future. Nevertheless, theoretical fluid dynamics is still largely a collection of flow examples and *Hydrodynamics* contains a wealth of them.

Two misprints in the text should be pointed out. In §697 the $\vec{z}$ component of the rotation vector should be 0 rather than $-\omega^2 z$. Another is in the footnote of §17 stating that the preface gives an explanation for the minus sign in the definition $\vec{u} = -\nabla\phi$ for the potential. Lamb's explanation, which was included in the fifth edition but omitted in the sixth, is that with this choice ϕ is the impulsive pressure, or the potential of an impulsive force, that would start the flow $\vec{u}$ from rest, rather than one that would stop the flow.

The value of Lamb's *Hydrodynamics* today is first as a storehouse of exact solutions for fluid dynamic problems, as stated earlier. In this aspect it is unequaled by modern texts. There are also certain fluid dynamic topics that are still best expressed in Lamb's book. An example is his discussion of the Hamiltonian formulation for fluid dynamic problems. In particular Chapter VI develops the Hamiltonian approach for the dynamics of solid particles in a fluid, treating the mixture as a single system. As Lamb points out at the end of the chapter, this approach has not been validated in all circumstances, and it seems to be fertile ground for further exploration. A related topic that is often omitted in contemporary texts is Clebsch coordinates, described in §167 at the end of Chapter VII.

Less tangible but equally important is the contact that *Hydrodynamics* provides with an earlier era of fluid dynamics. Lamb gave careful attribution to original sources, which is of great help to anyone interested in the history of fluid mechanics. More important to most readers is the perspective conveyed from a crucial period in the development of this subject. Since the final revision of *Hydrodynamics* in the 1930's, great progress has been made in fluid dynamics. Lamb's treatment of nonlinear water waves, shock waves, fluid dynamic stability, boundary layers and turbulence, for example, suggests many problems, of which a large number have since been solved but many others remain open. Thus *Hydrodynamics* provides us with a valuable measure of the past progress of fluid dynamics and with a compelling challenge for its future.

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